Mixed (and) Stabilised Finite Element Methods

Ramon Codina

Universitat Politècnica de Catalunya

EHF 2026, Paris

January 2026

Outline

Outline

Definition

Definition (mixed problem)

Consider a variational problem of the form: find $u \in V$ such that

$$B(u, v) = L(v) \quad \forall v \in V$$

with $B : V \times V \to \mathbb{R}$ bilinear and continuous, and $L : V \to \mathbb{R}$ continuous. We say that this is a **mixed problem** if $u = [u_1, u_2]$, $V = V_1 \times V_2$, with $V_1 \neq V_2$. The extension to n-unknowns is obvious.

Another way to write a mixed problem is: find $[u_1, u_2] \in V_1 \times V_2$ such that

$$\begin{aligned} a_{11}(u_1, v_1) + a_{12}(u_2, v_1) &= l_1(v_1) \quad \forall v_1 \in V_1 \\ a_{21}(u_1, v_2) + a_{22}(u_2, v_2) &= l_2(v_2) \quad \forall v_2 \in V_2 \end{aligned}$$

When V_1 and V_2 are spaces defined on geometrical domains of different dimension (e.g., d and $d-1$), the problem is called a **hybrid problem**.

Well-posedness

Suppose that $a_{ij} : V_j \times V_i \to \mathbb{R}$ is bilinear and continuous and $l_i : V_i \to \mathbb{R}$ linear and continuous. Let $u = [u_1, u_2]$, $V = V_1 \times V_2$ and

$$\begin{aligned} B([u_1, u_2], [v_1, v_2]) &= a_{11}(u_1, v_1) + a_{12}(u_2, v_1) + a_{21}(u_1, v_2) + a_{22}(u_2, v_2) \\ L([v_1, v_2]) &= l_1(v_1) + l_2(v_2) \end{aligned}$$

inf-sup Condition (elaborated later)

The problem is well-posed if, and only if,

$$\forall u \in V \quad \exists v \in V \mid B(u, v) \geq K_B \|u\|_V \|v\|_V,$$

for a constant $K_B > 0$. We will refer to this condition as the **inf-sup condition**.

Well-posedness for a system of equations

Let K_{ii} be the **coercivity constant** of a_{ii} and N_{ij} the **continuity constant** of a_{ij}, i.e.,

$$a_{ii}(u_i, u_i) \geq K_{ii} \|u_i\|^2_{V_i}, \quad a_{ij}(u_j, v_i) \leq N_{ij} \|u_j\|_{V_j} \|v_i\|_{V_i}$$

with the norms $\|\cdot\|_{V_1}, \|\cdot\|_{V_2}$ endowing V_1, V_2 of a Banach space structure.

Theorem

The inf-sup condition holds if

$$\det \begin{bmatrix} K_{11} & N_{12} \\ N_{21} & K_{22} \end{bmatrix} = K_{11} K_{22} - N_{12} N_{21} > 0$$

Proof of the well-posedness for systems

Given $u = [u_1, u_2] \in V$, let us pick $v = [u_1, \gamma u_2]$, with $\gamma > 0$. Then:

$$
\begin{aligned}
B(u,v) &= a_{11}(u_1,u_1) + a_{12}(u_2,u_1) + a_{21}(u_1,\gamma u_2) + a_{22}(u_2,\gamma u_2) \\
&\geq K_{11}\|u_1\|_{V_1}^2 - N_{12}\|u_1\|_{V_1}\|u_2\|_{V_2} - \gamma N_{21}\|u_1\|_{V_1}\|u_2\|_{V_2} + \gamma K_{22}\|u_2\|_{V_2}^2 \\
&\geq K_{11}\|u_1\|_{V_1}^2 - N_{12}\left(\frac{\beta_1}{2}\|u_1\|_{V_1}^2 + \frac{1}{2\beta_1}\|u_2\|_{V_2}^2\right) \\
&\quad + \gamma K_{22}\|u_2\|_{V_2}^2 - N_{21}\left(\frac{\beta_2}{2}\gamma^2\|u_2\|_{V_2}^2 + \frac{1}{2\beta_2}\|u_1\|_{V_1}^2\right) \\
&= \left(K_{11} - N_{12}\frac{\beta_1}{2} - N_{21}\frac{1}{2\beta_2}\right)\|u_1\|_{V_1}^2 \\
&\quad + \left(\gamma K_{22} - N_{21}\frac{\beta_2}{2}\gamma^2 - N_{12}\frac{1}{2\beta_1}\right)\|u_2\|_{V_2}^2
\end{aligned}
$$

Proof of the well-posedness for systems (cont)

B will be coercive if we can choose β_1, β_2, γ such that

$$K_{11} - N_{12}\frac{\beta_1}{2} - \frac{N_{21}}{2\beta_2} > 0 \Rightarrow \beta_1 < \frac{2K_{11}\alpha_1}{N_{12}}, \quad \beta_2^{-1} < \frac{2K_{11}\alpha_2}{N_{21}} \tag{1}$$

with $\alpha_1 + \alpha_2 = 1$ $(\alpha_1, \alpha_2 > 0)$ and

$$\gamma > A\gamma^2 + B, \quad A = \frac{N_{21}^2\beta_2}{4K_{11}K_{22}\alpha_2}, \quad B = \frac{N_{12}^2}{4K_{11}K_{22}\alpha_1} \tag{2}$$

This is possible to fulfil if

$$AB < \frac{1}{4} \iff N_{12}N_{21} < 2K_{11}K_{22}\sqrt{\alpha_1\alpha_2}$$

Since $\alpha_1 + \alpha_2 = 1$, the maximum of $\sqrt{\alpha_1\alpha_2}$ is $\frac{1}{2}$, and thus β_1, β_2 and γ satisfying (1) and (2) exist.

Saddle point problems

Very often $a_{22} = 0$ and $l_2 = 0$, i.e., the mixed problem is of the form:

$$
\begin{aligned}
a_{11}(u_1, v_1) + a_{12}(u_2, v_1) &= l_1(v_1) \quad \forall v_1 \in V_1 \\
a_{21}(u_1, v_2) &= 0 \qquad \forall v_2 \in V_2
\end{aligned}
$$

therefore, the simple sufficient condition presented above is not applicable. When a_{11} is symmetric and $a_{12}(v_2, v_1) = a_{21}(v_1, v_2)$, it is easily checked that the previous problem is equivalent to

$$
u_1 = \arg\inf \left\{ \frac{1}{2} a_{11}(v_1, v_1) - l_1(v_1) \right\} \text{ such that } a_{21}(u_1, v_2) = 0 \ \forall v_2 \in V_2
$$

i.e., it is **a saddle point problem**.

Finite element approximation

Let $V_{1,h} \subset V_1$, $V_{2,h} \subset V_2$ be FE spaces constructed from a partition $\{K\}$ of the computational domain Ω. We will see that, in general, the construction of $V_{1,h}$ and $V_{2,h}$ needs to be **different**. We consider the (Lagrangian) approximations:

$$v_i(x) \approx v_{i,h}(x) = \sum_{a=1}^{n_i} N_i^a(x) v_i^a, \quad i = 1, 2, \quad v_{i,h} \in V_{i,h}$$

For the moment, we may think of v_i as scalar functions. Then

$$\begin{aligned} a_{ij}(u_j, v_i) &= a_{ij}\left(\sum_{a=1}^{n_j} N_j^a u_j^a, \sum_{b=1}^{n_i} N_i^b v_i^b\right) = \sum_{a=1}^{n_j}\sum_{b=1}^{n_i} v_i^b \, a_{ij}(N_j^a, N_i^b)\, u_j^a \\ &= \mathbf{v}_i^T \mathbf{A}_{ij} \mathbf{u}_j \end{aligned}$$

where $\mathbf{u}_i^T = [u_i^1, u_i^2, \ldots, u_i^{n_i}]$, $\mathbf{A}_{ij}^{ba} = a_{ij}(N_j^a, N_i^b)$, $\mathbf{A}_{ij} \in \mathrm{Mat}(n_i, n_j)$.

Matrix structure

We may write

$$l_i(v_i) = \sum_{b=1}^{n_i} v_i^b \, l_i(N_i^b)$$

Introducing

$$\mathbf{f}_i^T = [l_i(N_i^1), \ldots, l_i(N_i^{n_i})]$$

we may write the final algebraic system as

$$[\boldsymbol{v}_1^T, \boldsymbol{v}_2^T] \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = [\boldsymbol{v}_1^T, \boldsymbol{v}_2^T] \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$$

or, since it must hold $\forall \boldsymbol{v}_1 \in \mathbb{R}^{n_1}, \boldsymbol{v}_2 \in \mathbb{R}^{n_2}$:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$$

In saddle point problems $\mathbf{A}_{22} = \mathbf{0}$.

Darcy's problem in primal form: Poisson's problem

Consider the following Poisson's problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\kappa\Delta u + \sigma u &= f \quad \text{in } \Omega \\ u &= \bar{u} \quad \text{on } \Gamma_u \\ -\kappa \frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma_g \end{aligned}$$

Let $V_s = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_u\}$. The weak form of the problem is: find $u : \Omega \to \mathbb{R}$ such that $u - \bar{u} \in V_s$ and

$$\begin{aligned} B(u, v) &= L(v) \quad \forall v \in V_s \\ B(u, v) &= \kappa \int_\Omega \nabla u \cdot \nabla v + \int_\Omega \sigma u v \\ L(v) &= \int_\Omega f v + \int_{\Gamma_g} v g \end{aligned}$$

Darcy's problem in primal form: weak form

We can also consider the mixed form of Poisson's problem, called **Darcy's problem**. It consists of finding $u : \Omega \to \mathbb{R}$ and $q : \Omega \to \mathbb{R}^d$ such that

$$\begin{aligned} \kappa^{-1} q + \nabla u &= 0 \quad \text{in } \Omega \\ \nabla \cdot q + \sigma u &= f \quad \text{in } \Omega \\ u &= \bar{u} \quad \text{on } \Gamma_u \\ -n \cdot q &= g \quad \text{on } \Gamma_g \end{aligned}$$

Observe that $q = -\kappa \nabla u$. For the **primal formulation** let V_s as for the irreducible form and $R_s = [L^2(\Omega)]^d$. Using the fact that

$$\int_\Omega v \nabla \cdot q = -\int_\Omega \nabla v \cdot q + \int_{\partial\Omega} v n \cdot q = -\int_\Omega \nabla v \cdot q + \int_{\Gamma_g} v(-g)$$

we may consider the following problem: find $[u, q] \in V_s \times R_s$ such that

$$\begin{aligned} \kappa^{-1}(q, r) + (\nabla u, r) = 0 \quad & \forall r \in R_s \\ -(\nabla v, q) + \sigma(v, u) = \langle v, f \rangle + \langle v, g \rangle_{\Gamma_g} \quad & \forall v \in V_s \end{aligned}$$

Darcy's problem in primal form: abstract formulation

Defining:

$$
\begin{aligned}
a : R_s \times R_s \to \mathbb{R}, &\quad q, r \mapsto a(q,r) = \kappa^{-1}(q,r) \\
b : V_s \times R_s \to \mathbb{R}, &\quad u, r \mapsto b(u,r) = (\nabla u, r) \\
c : V_s \times V_s \to \mathbb{R}, &\quad v, u \mapsto c(v,u) = \sigma(v,u) \\
l : V_s \to \mathbb{R}, &\quad v \mapsto l(v) = \langle v, f\rangle + \langle v, g\rangle_{\Gamma_g}
\end{aligned}
$$

we may write the weak form as:

$$
\begin{aligned}
a(q,r) + b(u,r) &= 0 \quad \forall r \in R_s \\
-b(v,q) + c(u,v) &= l(v) \quad \forall v \in V_s
\end{aligned}
$$

Let $U = [q,u]$, $V = [r,v]$ (note space for u) and

$$
\begin{aligned}
B(U,V) &= a(q,r) + b(u,r) - b(v,q) + c(u,v) \\
L(V) &= l(v)
\end{aligned}
$$

Darcy's problem in primal form: well posedness

Note that

$$B(U,U) = a(q,q) + c(u,u) = \kappa^{-1}\|q\|^2 + \sigma\|u\|^2$$

B is not coercive, since control on ∇u is missing. However, **the problem is well-posed**, as there holds:

$$\forall U \in R_s \times V_s \quad \exists V \in R_s \times V_s \text{ such that } B(U,V) \gtrsim \|U\|_{R_s\times V_s}\|V\|_{R_s\times V_s}$$

Even if this holds at the continuous level, **it does not necessarily hold at the discrete level**. The matrix structure of this discrete problem is:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{f} \end{bmatrix}$$

Note that, if $\sigma = 0$, $\mathbf{C} = \mathbf{0}$.

Darcy's problem in dual form: weak form

For the **dual formulation** let $V_s = L^2(\Omega)$ and $R_s = H(\mathrm{div}, \Omega)$. Since

$$\int_\Omega r \cdot \nabla u = -\int_\Omega \nabla \cdot r\, u + \int_{\partial\Omega} n \cdot r\, u = -\int_\Omega \nabla \cdot r\, u + \int_{\Gamma_u} n \cdot r\, \bar{u}$$

we may consider the following problem: find $[u, q] \in V_s \times R_s$ such that

$$\kappa^{-1}(q, r) - (u, \nabla \cdot r) = -\langle n \cdot r, \bar{u} \rangle_{\Gamma_u} \quad \forall r \in R_s$$
$$(v, \nabla \cdot q) + \sigma(v, u) = \langle v, f \rangle \quad \forall v \in V_s$$

- The structure of the problem is the same as for the primal formulation, with a **a different functional setting**.
- The inf-sup condition holds.
- Discrete inf-sup stable finite element spaces for the primal and the dual formulation **are different**.

Stokes problem

It consists of finding $u : \Omega \to \mathbb{R}^d$, $p : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nu \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \quad \text{(for simplicity)} \end{aligned}$$

Let $V = H_0^1(\Omega)^d$, $Q = L^2(\Omega)/\mathbb{R}$. The weak form is: find $u \in V$, $p \in Q$ s.t.

$$\begin{aligned} \nu(\nabla u, \nabla v) - (p, \nabla \cdot v) &= \langle f, v \rangle \quad \forall v \in V \\ (q, \nabla \cdot u) &= 0 \quad \forall q \in Q \end{aligned}$$

The matrix structure of the FE approximation to this problem is

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

V and Q satisfy the inf-sup condition. However, when FE spaces $V_h \subset V$ and $Q_h \subset Q$ are considered, this condition is **not obvious to be satisfied**.

Maxwell's problem: problem statement

It consists of finding $u : \Omega \to \mathbb{R}^d$ such that $\nabla \cdot u = 0$ and

$$\nabla \times \nabla \times u = f \quad \text{in } \Omega \quad (\text{with } \nabla \cdot f = 0)$$
$$n \times u = 0 \quad \text{on } \partial\Omega \quad (\text{for simplicity})$$

The curl-curl operator is not injective ($\nabla \times \nabla\varphi = 0$ for any φ, for example), and one can make the problem well-posed in several ways. One is Kikuchi's formulation, which consists of finding $u : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$ s.t.:

$$\begin{aligned} \nabla \times \nabla \times u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \\ n \times u &= 0 \quad \text{on } \partial\Omega \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

The continuous solution is $p = 0$, since p solves the problem:

$$\begin{aligned} \Delta p &= 0 \quad \text{in } \Omega \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Maxwell's problem: functional setting and FE approximation

Let us introduce the space:

$$
\begin{aligned}
V &= H_0(\text{curl};\Omega) \\
&= \{v : \Omega \to \mathbb{R}^d \mid v \in L^2(\Omega)^d, \nabla \times v \in L^2(\Omega)^d, n \times v = 0 \text{ on } \partial\Omega\}
\end{aligned}
$$

If $Q = H_0^1(\Omega)$, the weak form of the problem is: find $u \in V$ and $p \in Q$ s.t.

$$
\begin{aligned}
(\nabla \times u, \nabla \times v) + (v, \nabla p) = (v, f) \quad \forall v \in V \\
-(u, \nabla q) = 0 \quad \forall q \in Q
\end{aligned}
$$

The matrix structure of the FE approximation is:

$$
\begin{bmatrix} \mathbf{A} & \mathbf{G} \\ -\mathbf{G}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}
$$

It is a saddle point problem.

Elasticity: problem statement

The equations modeling a linear elastic body are the following:

$$
\begin{aligned}
-\nabla \cdot \sigma &= f \quad \text{in } \Omega \quad \text{(Cauchy's equation)} \\
C^{-1} : \sigma - \varepsilon &= 0 \quad \text{in } \Omega \quad \text{(Constitutive equation)} \\
\varepsilon - \nabla^s u &= 0 \quad \text{in } \Omega \quad \left(\text{Geometrical equation, } \nabla^s u = \frac{1}{2}(\nabla u^T + \nabla u)\right) \\
u &= 0 \quad \text{on } \partial\Omega \quad \text{(for simplicity)}
\end{aligned}
$$

Where:

- $\sigma : \Omega \to \mathbb{R}^d \otimes \mathbb{R}^d$: Cauchy stress tensor.
- $\varepsilon : \Omega \to \mathbb{R}^d \otimes \mathbb{R}^d$: Strain tensor (infinitesimal strain).
- $u : \Omega \to \mathbb{R}^d$: Displacement field.
- $C \in \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d$: Constitutive tensor.

Elasticity: incompressible materials

There holds:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$$
$$\varepsilon : C : \varepsilon = \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \gtrsim E \|\varepsilon\|^2$$

This makes the problem **elliptic**.
If the material is incompressible, it is convenient to decompose:

$$\sigma = -pI + \sigma', \quad p = -\frac{1}{3}\mathrm{tr}(\sigma), \quad \mathrm{tr}(\sigma') = 0$$

σ' is **deviatoric** and pI is **volumetric**. In this case, the equations are:

$$-\nabla \cdot \sigma' + \nabla p = f$$
$$C_{\mathrm{dev}}^{-1} : \sigma' - \varepsilon' = 0$$
$$\varepsilon' - \mathrm{dev}(\nabla^s u) = 0$$
$$\nabla \cdot u = 0$$

Elasticity: weak form and FE approximation

Considering all possible variables, the weak form would is: find $\sigma \in V_\sigma, \varepsilon \in V_\varepsilon, p \in V_p$ and $u \in V_u$ such that

$$
\begin{aligned}
(\nabla^s v, \sigma') - (p, \nabla \cdot v) &= \langle f, v \rangle \quad \forall v \in V_u \\
(\tau, \sigma') - (\tau, C_{\text{dev}} : \varepsilon') &= 0 \quad \forall \tau \in V_\varepsilon \\
(\eta, \varepsilon') - (\eta, \nabla^s u) &= 0 \quad \forall \eta \in V_\sigma \\
(q, \nabla \cdot u) &= 0 \quad \forall q \in V_p \quad \text{(incompressible case)}
\end{aligned}
$$

with $V_u = [H_0^1(\Omega)]^d$, $V_\sigma = [L^2(\Omega)]^{d\times d}$, $V_\varepsilon = [L^2(\Omega)]^{d\times d}$, $V_p = L^2(\Omega)/\mathbb{R}$.
The matrix structure of the FE approximation is:

$$
\begin{bmatrix} -\mathbf{M}_c & \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{M} & \mathbf{0} & -\mathbf{G}_s & \mathbf{0} \\ \mathbf{0} & -\mathbf{G}_s^T & \mathbf{0} & \mathbf{G} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}^T & \mathbf{0} \end{bmatrix}
\begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\sigma} \\ \mathbf{u} \\ \mathbf{p} \end{bmatrix}
=
\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{f} \\ \mathbf{0} \end{bmatrix}
$$

Elasticity: σ-u formulation

A particular case of interest is the σ-u formulation, which reads:

$$-\nabla \cdot \sigma = f$$
$$C^{-1} : \sigma - \nabla^s u = 0$$

The mathematical structure is very similar to that of Darcy's problem. It also admits **a primal and a dual formulation**. For the dual formulation, the weak form is

$$-(\nabla \cdot \sigma, v) = \langle f, v \rangle \quad \forall v \in V_u = [L^2(\Omega)]^d$$
$$(\tau, C^{-1} : \sigma) + (\nabla \cdot \tau, u) = 0 \quad \forall \tau \in V_\sigma = [H(\mathrm{div}, \Omega)]^d \quad \text{(zero BCs)}$$

The matrix structure of the FE approximation is:

$$\begin{bmatrix} \mathbf{M}_c & -\mathbf{G}_s \\ -\mathbf{G}_s^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}$$

inf-sup stable elements for this problem are quite involved.

Reissner-Mindlin plates: problem statement

The bending of a plate using Reissner-Mindlin theory yields the following BVP: find $\theta : \Omega \to \mathbb{R}^2$ and $w : \Omega \to \mathbb{R}$ (rotations and deflection) s.t.:

$$
\begin{aligned}
-k_1 \Delta \theta - k_2 \nabla(\nabla \cdot \theta) - \frac{1}{\varepsilon}(\nabla w - \theta) &= m \quad \text{in } \Omega \\
-\frac{1}{\varepsilon} \nabla \cdot (\nabla w - \theta) &= q \quad \text{in } \Omega \\
w &= 0 \quad \text{on } \partial\Omega \\
\theta &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

where

$$
k_1 = \frac{Et^3}{24(1+\nu)}, \qquad k_2 = \frac{Et^3}{24(1-\nu^2)}
$$

$$
\varepsilon = \frac{2(1+\nu)}{E\kappa t}, \qquad E, \nu : \ \text{elastic parameters}
$$

t : plate thickness, $\qquad \kappa$: shear correction factor

Reissner-Mindlin plates: weak form

The weak form is: find $\theta \in V_\theta^2 = H_0^1(\Omega)^2, w \in V_w = H_0^1(\Omega)$ s.t.

$$k_1(\nabla\theta, \nabla\psi) + k_2(\nabla\cdot\theta, \nabla\cdot\psi) + \frac{1}{\varepsilon}(\nabla w - \theta, \nabla v - \psi) = (\psi, m) + (v, q)$$

$\forall\psi \in V_\theta^2, \forall v \in V_w$.

- Since $V_\theta = V_w$, in principle the problem **is not a mixed problem**.
- It turns out that $k_i = O(t^3)$, $i = 1, 2$, and $\varepsilon^{-1} = O(t)$.
- In the limit $t \to 0$, the solution should be $\theta = \nabla w$, and therefore $V_w = H_0^1(\Omega) \cap H^2(\Omega)$.
- Since the functional framework changes when $t \to 0$, this is an example of **singularly perturbed problem**.
- When FE spaces are used, it is convenient to consider $V_{\theta,h} \subset V_\theta$, $V_{w,h} \subset V_w$, with $V_{\theta,h} \neq V_{w,h}$ to allow for the limit $t \to 0$, i.e., the problem **needs to be treated as a mixed problem**.

Reissner-Mindlin plates: FE approximation

The matrix form of the FEA is:

$$\begin{bmatrix} k_1\mathbf{K}_1 + k_2\mathbf{K}_2 + \frac{1}{\varepsilon}\mathbf{M} & -\frac{1}{\varepsilon}\mathbf{G} \\ -\frac{1}{\varepsilon}\mathbf{G}^T & \frac{1}{\varepsilon}\mathbf{L} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{m} \\ \mathbf{q} \end{bmatrix}$$

If matrix

$$\frac{1}{\varepsilon} \begin{bmatrix} \mathbf{M} & -\mathbf{G} \\ -\mathbf{G}^T & \mathbf{L} \end{bmatrix}$$

is **not singular** and the FE spaces **are not such that** $\nabla V_{w,h} \subset V_{\theta,h}$**, the solution suffers from shear locking** when $t \to 0$, i.e., $\boldsymbol{\theta} \to \mathbf{0}$ and $\mathbf{w} \to \mathbf{0}$ as $t \to 0$.

Outline

Main result

The problem we consider is: find $U \in X_U$ such that

$$B(U,V) = L(V) \quad \forall V \in X_V$$

where $U^T = [u_1, \dots, u_n]$, $V^T = [v_1, \dots, v_n]$, and B and L are continuous, but B is not necessarily coercive.

Theorem (Babuška-Lax-Milgram)

The problem is well posed if, and only if,

1. $$\inf_{U \in X_U \setminus \{0\}} \sup_{V \in X_V \setminus \{0\}} \frac{|B(U,V)|}{\|U\|_{X_U} \|V\|_{X_V}} \geq K_B > 0$$
2. $$\sup_{U \in X_U} \frac{|B(U,V)|}{\|V\|_{X_V}} > 0 \quad \forall V \in X_V \setminus \{0\}$$

If these two conditions hold, the unique solution can be bounded as

$$\|U\|_{X_U} \leq \frac{N_L}{K_B}$$

Proof of Babuška-Lax-Milgram's Theorem

There holds

$$\inf_U \sup_V \frac{|B(U,V)|}{\|U\|\|V\|} \geq K_B > 0 \Leftrightarrow \forall U \in X_u \ \sup_V \frac{|B(U,V)|}{\|V\|} \geq K_B \|U\|$$
$$\Leftrightarrow \forall U \in X_U \ \exists V \in X_V \text{ such that } B(U,V) \geq K_B \|U\|_{X_U} \|V\|_{X_V}$$

If this and the non-degeneracy condition hold, the closed range theorem implies that the problem is well posed. The stability bound for the solution is trivial:

$$K_B \|U\|\|V\| \leq B(U,V) = L(V) \leq N_L \|V\|$$

Remark In the finite dimensional case, the inf-sup condition is

$$\min_U \max_V \frac{V^t B U}{\|U\|\|V\|} \geq K_B > 0$$

It is seen that k_B is an estimate for the minimum generalised eigenvalue of B. Thus, the inf-sup condition is nothing but the invertibility of B.

Generalised Céa's Lemma

Suppose that $X_U = X_V = X$ is Hilbert and separable:

$$X = \mathrm{span}\{\phi_n\}_{n=1}^{\infty}$$

and consider $X_N = \mathrm{span}\{\phi_n\}_{n=1,\dots,N<\infty}$. We may then analyse the finite dimensional problem: find $U_N \in X_N$ such that

$$B(U_N, V_N) = L(V_N) \quad \forall V_N \in X_N$$

Theorem (Generalised Céa's Lemma)

If the inf-sup condition

$$\forall U_N \in X_N \; \exists V_N \in X_N \mid B(U_N, V_N) \geq K_B \|U_N\| \|V_N\|$$

holds, then the finite dimensional problem has a unique solution that satisfies:

$$\|U - U_N\| \leq \left(1 + \frac{N_B}{K_B}\right) \inf_{W_N \in X_N} \|U - W_N\|$$

Proof of the Generalised Céa's Lemma

Well posedness is trivial. Observe also that $\forall V_N \in X_N$:

$$K_B \|U_N\| \|V_N\| \leq B(U_N, V_N) = L(V_N) \leq N_L \|V_N\| \Rightarrow \|U_N\| \leq \frac{N_L}{K_B}$$

We also have that:

$$\begin{aligned} B(U, V_N) &= L(V_N) \quad \forall V_N \in X_N \\ B(U_N, V_N) &= L(V_N) \quad \forall V_N \in X_N \end{aligned}$$

$\Rightarrow B(U - U_N, V_N) = 0 \; \forall V_N \in X_N$ (consistency). Then $\forall W_N \in X_N \; \exists V_N \in X_N$ such that:

$$\begin{aligned} K_B \|U_N - W_N\| \|V_N\| &\leq B(U_N - W_N, V_N) + B(U - U_N, V_N) \\ &= B(U - W_N, V_N) \leq N_B \|U - W_N\| \|V_N\| \end{aligned}$$

$\Rightarrow \|U_N - W_N\| \leq \frac{N_B}{K_B} \|U - W_N\|$. Using the triangle inequality:

$$\begin{aligned} \|U - U_N\| &\leq \|U - W_N\| + \|W_N - U_N\| \\ &\leq \left(1 + \frac{N_B}{K_B}\right) \|U - W_N\| \end{aligned}$$

Application to saddle point problems

The two previous results are general for any BVP. Consider now:

$$a(u,v) + b(p,v) = l_u(v)$$
$$-b(q,u) + c(p,q) = l_p(q)$$

with $U = [u,p]$, $V = [v,q]$, $X = V \times Q$, all forms continuous, and a and c symmetric. This problem is equivalent to (problem SP):

$$B([u,p],[v,q]) = a(u,v) + b(p,v) - b(q,u) + c(p,q)$$
$$= L([v,q]) = l_u(v) + l_p(q)$$

Theorem (Saddle point problem structure)

Suppose that $a(v,v) > 0 \; \forall v \in V \setminus \; \{0\}$ *and* $c(p,p) \geq 0$. *Then:*

$$[u,p] = \arg \inf_{v \in V} \sup_{q \in Q} \mathcal{L}(v,q)$$

where $\mathcal{L}(v,q) = \frac{1}{2}a(v,v) + b(q,v) - \frac{1}{2}c(q,q) - l_u(v) - l_p(q)$.

Proof of the saddle point problem structure

Let $f(\epsilon_1, \epsilon_2) = \mathcal{L}(u + \epsilon_1 v, p + \epsilon_2 q)$.

$$\left.\frac{\partial f}{\partial \epsilon_1}\right|_{\epsilon_1=\epsilon_2=0} = a(u,v) + b(p,v) - l_u(v) = 0$$
$$\left.\frac{\partial f}{\partial \epsilon_2}\right|_{\epsilon_1=\epsilon_2=0} = b(q,u) - c(p,q) - l_p(q) = 0$$

so that if $[u,p]$ solves SP it is a critical point of $\mathcal{L}$. On the other hand:

$$\det \begin{bmatrix} \frac{\partial^2 f}{\partial \epsilon_1^2} - \lambda & \frac{\partial^2 f}{\partial \epsilon_1 \partial \epsilon_2} \\ \frac{\partial^2 f}{\partial \epsilon_1 \partial \epsilon_2} & \frac{\partial^2 f}{\partial \epsilon_2^2} - \lambda \end{bmatrix} = \det \begin{bmatrix} a(v,v) - \lambda & b(q,v) \\ b(q,v) & -c(q,q) - \lambda \end{bmatrix} = 0$$

$\Rightarrow 2\lambda_{1,2} = a(v,v) - c(q,q) \pm \sqrt{(a(v,v) + c(q,q))^2 + 4b(q,v)^2}$.
$\lambda_1 \geq a(v,v) > 0$ and $\lambda_2 \leq -c(q,q) \leq 0$, which indicates a saddle point structure. $\mathcal{L}(v,0) = \frac{1}{2}a(v,v) - l_u(v)$ clearly has a minimum as a critical point, since $a(v,v) > 0$.

Ladyzhenskaya-Babuška-Brezzi's Theorem

Theorem (Ladyzhenskaya-Babuška-Brezzi)

For a **saddle point problem**, *suppose that:*

H1. a *and* b *are continuous,* a *is non-negative* $(a(v,v) \geq 0, \forall v \in V)$.

H2. $a(u,v)$ *is coercive in* $K = \{v \in V \mid b(q,v) = 0 \ \forall q \in Q\}$*. i.e.,* $\exists K_a > 0$ *such that* $a(u,u) \geq K_a \|u\|^2 \ \forall u \in K$.

H3. *"Little" inf-sup condition:* $\forall p \in Q \ \exists v \in V \mid b(p,v) \geq K_b \|p\| \|v\|$.

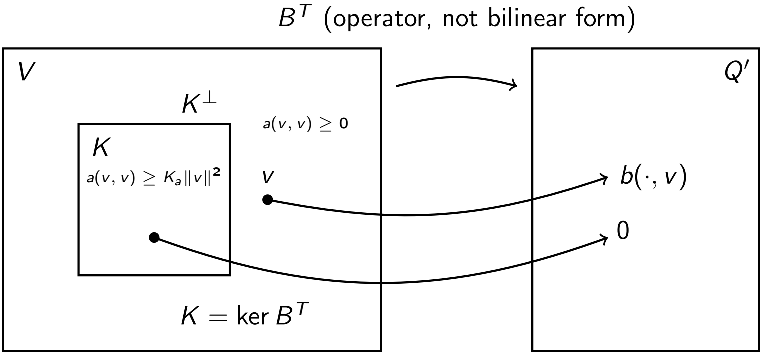
H4. $c(\cdot,\cdot)$ *is continuous and coercive in* $\|\cdot\|_{\hat{Q}}$*, possibly weaker than* $\|\cdot\|_Q$*:* $\exists N_c > 0$, $K_c = \gamma N_c$, $\gamma > 0$ *such that*

$$c(p,q) \leq N_c \|p\|_{\hat{Q}} \|q\|_{\hat{Q}} \quad \forall p,q \in Q$$
$$c(p,p) \geq K_c \|p\|_{\hat{Q}}^2 \quad \forall p \in Q$$

Then, $B([u,p],[v,q])$ *satisfies the inf-sup condition in the norm*

$$\|[v,q]\|_{V\times Q} = \sqrt{K_a}\|v\|_V + \sqrt{M_{ab}}\|q\|_Q \text{ for a certain constant } M_{ab} > 0.$$

Proof of LBB's Theorem: setting



Proof of LBB's Theorem (I)

$B(\cdot,\cdot)$ is trivially continuous. Let B^T be the operator $b(\cdot,v)$ and $K = \text{Ker } B^T$. Let $V = K \oplus K^\perp$, with $\perp$ with respect to the inner product that restricted to K coincides with $a(\cdot,\cdot)$. Then, if $v = v_0 + v_1$, $v_0 \in K, v_1 \in K^\perp$:

$$a(v,v) = a(v_0,v_0) + a(v_1,v_1)$$

There holds:

$$B([u,p],[u_0,0]) = a(u,u_0) = a(u_0,u_0) \geq K_a \|u_0\|^2 \tag{3}$$

The closed range theorem implies:

$$\forall v \in K^\perp \;\exists\, q \in Q \text{ s.t. } K_b \|q\|_Q \|v\|_V \leq b(q,v)$$

(In the finite dimensional case, this is equivalent to saying that the rank of a matrix is equal to the rank of the transpose and the generalised eigenvalues are the same).

Proof of LBB's Theorem (II)

Thus, for $v = u_1 \in K^\perp$ $\exists\, q_1 \in Q$ such that $K_b \|q_1\| \|u_1\|_V \leq b(q_1, u_1)$. We may take $\|q_1\|_Q = \frac{K_a}{K_b}\|u_1\|_V$. Then:

$$
\begin{aligned}
B([u,p],[0,q_1]) &= -b(q_1,u) + c(p,q_1) \\
&\geq K_b \|q_1\| \|u_1\| - N_c \|p\|_{\hat{Q}} \|q_1\|_{\hat{Q}} \\
&\geq K_a \|u_1\|^2 - \left(\frac{1}{2\alpha}\|p\|_{\hat{Q}}^2 + \frac{\alpha}{2}\|u_1\|^2 \right) \frac{N_c K_a}{K_b} \\
&\geq \frac{K_a}{2}\|u_1\|^2 - \frac{N_c^2 K_a}{2K_b^2}\|p\|_{\hat{Q}}^2 \quad \text{for } \alpha = \frac{K_b}{N_c} \text{ if } N_c \neq 0 \qquad (4)
\end{aligned}
$$

Given $p \in Q$, there exists $v_p \in V$ such that $K_b \|p\| \|v_p\| \leq b(p, v_p)$. Let us take $\|v_p\| = \frac{K_a K_b}{N_a^2}\|p\|_Q$. We have that

$$
\begin{aligned}
B([u,p],[v_p,0]) &= a(u,v_p) + b(p,v_p) \geq K_b \|p\| \|v_p\| - N_a \|u\| \|v_p\| \\
&\geq K_b \|p\| \|v_p\| - \frac{N_a}{2\alpha}\|u\|^2 - \frac{N_a \alpha}{2}\|v_p\|^2 \\
&\geq \frac{K_a K_b^2}{2N_a^2}\|p\|^2 - \frac{K_a}{2}\|u\|^2 \quad \text{for } \alpha = \frac{N_a}{K_a} \qquad (5)
\end{aligned}
$$

Proof of LBB's Theorem (III)

Finally:

$$B([u,p],[u,p]) = a(u,u) + c(p,p) \geq K_c \|p\|_{\hat{Q}}^2 \qquad (6)$$

The element we were looking for is:

$$[v,q] = [u_0 + v_p + \beta u, 2q_1 + \beta p]$$

Summing (3)-(6):

$$B([u,p],[v,q]) \geq K_a \|u\|_V^2 + \frac{K_a K_b^2}{N_a^2} \|p\|_Q^2 + \left(\beta \gamma N_c - \frac{N_c^2 K_a}{K_b^2} \right) \|p\|_{\hat{Q}}^2$$

For β large enough, it follows that

$$B([u,p],[v,q]) \gtrsim \|[u,p]\|_{V \times Q}^2$$

It is readily checked that $\|[v,q]\|_{V\times Q} \lesssim \|[u,p]\|_{V\times Q}$ from where the theorem follows.

FE approximation of saddle point problems

Let us consider the FEA: find $u_h \in V_h \subset V$ and $p_h \in Q_h \subset Q$ s.t.

$$\begin{aligned} a(u_h, v_h) + b(p_h, v_h) &= l_u(v_h) \quad \forall v_h \in V_h \\ -b(q_h, u_h) + c(p_h, q_h) &= l_p(q_h) \quad \forall q_h \in Q_h \end{aligned}$$

The assumptions of the LBB theorem need to apply. In particular:

$$\inf_{p_h \in Q_h} \sup_{v_h \in V_h} \frac{b(p_h, v_h)}{\|p_h\|_Q \|v_h\|_V} \geq \beta > 0$$

This condition **is not inherited for the discrete problem**. Obviously:

$$\forall p_h \in Q_h \; \exists \, v \in V \text{ s.t. } b(p_h, v) \geq \beta \|p_h\|_Q \|v\|_V \quad \text{but maybe } v \notin V_h$$

Main remark

In some cases, the discrete inf-sup condition is difficult or inconvenient to satisfy. For example, it prevents from using equal interpolation for u_h, p_h. A possibility to avoid this is to use **stabilised finite element methods**.

Optimisation problem

The matrix structure of the problem is

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_p \end{bmatrix}$$

If $\mathbf{C} = \mathbf{0}$ and $\mathbf{f}_p = \mathbf{0}$, $\mathbf{p}$ can be understood as a Lagrange multiplier to impose the condition $\mathbf{B}^T\mathbf{u} = \mathbf{0}$. Indeed, the problem corresponds to the minimization of the functional

$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{A}\mathbf{u} - \mathbf{u}^T\mathbf{f}_u$$

under the restriction $\mathbf{B}^T\mathbf{u} = \mathbf{0}$, which is equivalent to optimise the Lagrangian

$$\mathcal{L}(\mathbf{u}, \mathbf{p}) = \frac{1}{2}\mathbf{u}^T\mathbf{A}\mathbf{u} - \mathbf{u}^T\mathbf{f}_u + \mathbf{p}^T\mathbf{B}^T\mathbf{u}$$

The Euler-Lagrange equations of this functional are precisely the equations to be solved.

Outline

Problem to be solved

Let us consider a quite general second order BVP of the form:

$$\mathcal{L}u = f \quad , \quad u : \Omega \to \mathbb{R}^n$$

where the operator $\mathcal{L}$ is defined as:

$$\mathcal{L}u = -\partial_i(K_{ij}\partial_j u) + A_{c,i}\partial_i u + A_{f,i}\partial_i u + Su$$

where $A_{f,i}, A_{c,i}, S, K_{ij} \in \mathrm{Mat}(n, n)$. As BCs, we consider:

$$\mathcal{D}u = 0 \quad \text{on } \partial\Omega$$

where $\mathcal{D} : \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$, depends on which components of u can be prescribed. Matrices $\{K_{ij}\}_{i,j=1,d}$ are assumed to define a positive semi-definite quadratic form. The convection matrices $A_i = A_{c,i} + A_{f,i}$ are split into those that will not be integrated by parts ($A_{c,i}$) and those that will ($A_{f,i}$). The reaction matrix S is assumed to be positive semi-definite.

Adjoint operators

In any region $\omega \subseteq \Omega$, let us introduce:

$$B_\omega(u,v) = (\partial_i v, K_{ij}\partial_j u)_\omega + (v, A_{c,i}\partial_i u)_\omega - (\partial_i(A^t_{f,i}v), u)_\omega + (v, Su)_\omega$$

Defining the operators:

- Adjoint operator: $\mathcal{L}^* v = -\partial_i(K^t_{ij}\partial_j v) - \partial_i(A^t_{c,i}v) - \partial_i(A^t_{f,i}v) + S^t v$
- Flux operator: $\mathcal{F}_n u = n_i K_{ij}\partial_j u - n_i A_{f,i} u$
- Adjoint flux: $\mathcal{F}^*_n v = n_i K^t_{ij}\partial_j v + n_i A^t_{c,i} v$

there holds:

$$\begin{aligned} B_\omega(u,v) &= (\mathcal{L}u, v)_\omega + (\mathcal{F}_n u, \mathcal{D}v)_{\partial\omega} \\ &= (u, \mathcal{L}^* v)_\omega + (\mathcal{D}u, \mathcal{F}^*_n v)_{\partial\omega} \end{aligned}$$

Sources of numerical instability

Calling $L(v) = (v, f)$, $B_\Omega(u, v) \equiv B(u, v)$, the problem we consider is: find $u \in X$ such that:

$$B(u, v) = L(v) \quad \forall v \in X$$

where $u = [u_1, \ldots, u_n] \in X = X_1 \times \cdots \times X_n$. The FEA to this problem may suffer from two fundamental sources of instability:

1. Lack of compatibility between $X_{h,1}, \ldots, X_{h,n}$; i.e., failure to satisfy the appropriate inf-sup conditions. This is the classical situation in mixed methods.
2. Singular perturbation problems, i.e., $\mathcal{L}$ depends on parameters that, in the limit, change the functional framework of the problem. Thus, $u \in X$, but in the limit $u \in \tilde{X} \neq X$. This is the situation in convection-diffusion problems or in plates with thickness $t \to 0$.

Stabilised FE methods

Consider the Galerkin FEA: find $u_h \in X_h \subset X$ s.t.

$$B(u_h, v_h) = L(v_h) \quad \forall v_h \in X_h$$

Stabilised FEMs consist in replacing this problem by: find $u_h \in X_h$ such that:

$$\begin{aligned} B_h(u_h, v_h) &= B(u_h, v_h) + B_s(u_h, v_h) \\ &= L_h(v_h) =: L(v_h) + L_s(v_h) \quad \forall v_h \in V_h \end{aligned}$$

where $B_s(u_h, v_h)$ and $L_s(v_h)$ are usually mesh dependent. The discrete problem has to satisfy two main conditions:

1. It should be consistent or, at least, weakly consistent:

$$|B_h(u, v_h) - L_h(v_h)| \leq Ch^r, \quad r \text{ large enough}$$

2. It should satisfy an inf-sup condition in a mesh dependent norm:

$$\forall u_h \in X_h \; \exists v_h \in X_h \mid B_h(u_h, v_h) \geq K_{B_h} \|u_h\|_{h,X} \|v_h\|_{h,X}$$

Variational multiscale (VMS) approach

There are several methods to design stabilised FEMs. Here we shall follow the VMS approach. The basic idea is to split:

$$\begin{aligned} X &= X_h \oplus X', \quad X' \text{ to be determined} \\ u &= u_h + u' \end{aligned}$$

The original continuous problem is equivalent to:

$$\begin{cases} B(u_h, v_h) + B(u', v_h) = L(v_h) & \forall v_h \in X_h \\ B(u_h, v') + B(u', v') = L(v') & \forall v' \in X' \end{cases}$$

u' is called sub-grid scale (SGS).

Basic identities

Thinking that the approximation to u' may be discontinuous, and that we will approximate u', but not its derivatives, we may write:

$$B(u', v_h) = \sum_K B_K(u', v_h)$$
$$= \sum_K \left[(u', \mathcal{L}^* v_h)_K + (\mathcal{D}u', \mathcal{F}_n^* v_h)_{\partial K} \right]$$

$$B(u_h, v') + B(u', v') = \sum_K [B_K(u_h, v') + B_K(u', v')]$$
$$= \sum_K [(\mathcal{L}u_h, v')_K + (\mathcal{F}_n u_h, \mathcal{D}v')_{\partial K} + (u', \mathcal{L}^* v')_K + (\mathcal{F}_n u', \mathcal{D}v')_{\partial K}]$$
$$= \sum_K [(\mathcal{L}u_h, v')_K + (\mathcal{L}u', v')_K] = \sum_K (v', f)$$

assuming that $\mathcal{F}_n u_h + \mathcal{F}_n u'$ is continuous and so is $\mathcal{D}v'$.

Split continuous problem

The original problem can be written as:

Continuous problem for the FEM solution and the SGS

$$B(u_h, v_h) + \sum_K \left[(u', \mathcal{L}^* v_h)_K + (\mathcal{D}u', \mathcal{F}_n^* v_h)_{\partial K} \right] = L(v_h)$$

$$\sum_K (\mathcal{L}u' + \mathcal{L}u_h, v')_K = \sum_K (v', f)_K$$

Calling $\mathcal{R}u_h = f - \mathcal{L}u_h$ the finite element residual in each K, the equation for the SGS is:

$$\sum_K (\mathcal{L}u', v')_K = \sum_K (v', \mathcal{R}u_h)_K$$

Approximation of the SGSs on the element edges

We have assumed that $\mathcal{F}_n u_h + \mathcal{F}_n u'$ is continuous, but u' on the element edge must be such that this is possible. Several heuristic arguments lead to the approximation:

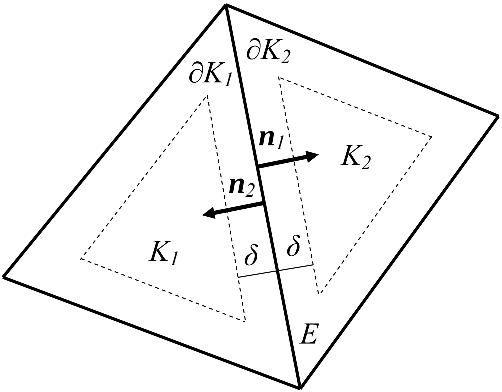
$$\mathcal{D}u'|_E \approx \tau_E [\![\mathcal{F}_n u_h]\!]_E \quad , \quad \tau_E \in \mathrm{Mat}(n,n)$$

where $[\![\mathcal{F}_n u_h]\!]_E = \mathcal{F}_{n_1} u_h|_E + \mathcal{F}_{n_2} u_h|_E$ being $n_1 = -n_2$. This leads to:

$$\begin{aligned}\sum_K (\mathcal{D}u', \mathcal{F}_n^* v_h)_{\partial K} &= \sum_E (\mathcal{D}u', [\![\mathcal{F}_n^* v_h]\!])_E \\ &= \sum_E (\tau_E [\![\mathcal{F}_n u_h]\!], [\![\mathcal{F}_n^* v_h]\!])_E\end{aligned}$$

The jump of the flux on the edges of $\partial\Omega$ is just this flux. If there are Neumann-type BC's on part of $\partial\Omega$, of the form $\mathcal{F}_n u = g$, the jump of the edges on this boundary has to be replaced by $\mathcal{F}_n u_h - g$.

Approximation of the SGSs on the element edges: notation



Approximation of the SGSs in the element interiors

Several heuristic arguments (bubbles, Green's function approximation, approximate Fourier analysis) lead to the following fundamental approximation:

$$\sum_K (\mathcal{L}u', v')_K \approx \sum_K (\tau_K^{-1} u', v')_K$$

where $\tau_K \in \text{Mat}(n, n)$ is the so-called **matrix of stabilisation parameters**. Therefore, the equation for the SGSs in the element interiors is:

$$\sum_K (\tau_K^{-1} u', v')_K = \sum_K (v', f - \mathcal{L}u_h)_K$$

If P' is the L^2 projection onto the space V', we have that:

$$u'|_K = \tau_K P'(f - \mathcal{L}u_h)|_K$$

If V' is taken as the space of FE residuals, then $P' = I$. This is the most common approach in the literature. P' can also be the projection onto a bubble space. If $V' = V_h^{\perp}$, then $P' = P_h^{\perp} = I - P_h$, leading to the Orthogonal Sub-Grid Scale (OSGS) formulation.

Final stabilised FE problem

Stabilised FE problem

Find $u_h \in V_h$ such that:

$$B(u_h, v_h) + \sum_K (\tau_K P'(f - \mathcal{L}u_h), \mathcal{L}^* v_h)_K + \sum_E (\tau_E [\![\mathcal{F}_n u_h]\!], [\![\mathcal{F}_n^* v_h]\!])_E = L(v_h) \quad \forall v_h \in V_h$$

In terms of the general structure of stabilised FEMs:

$$B_s(u_h, v_h) = \sum_K (\tau_K P'(-\mathcal{L}u_h), \mathcal{L}^* v_h)_K + \sum_E (\tau_E [\![\mathcal{F}_n u_h]\!], [\![\mathcal{F}_n^* v_h]\!])_E$$

$$L_s(v_h) = \sum_K (\tau_K P'(-f), \mathcal{L}^* v_h)_K$$

Outline

Model problems

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. The problems we are interested in consist in finding $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and $p : \Omega \longrightarrow \mathbb{R}$ such that

Stokes:

$$-\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f},$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

Maxwell:

$$\lambda \nabla \times \nabla \times \boldsymbol{u} + \nabla p = \boldsymbol{f},$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

Darcy:

$$\sigma \boldsymbol{u} + \nabla p = \boldsymbol{f},$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

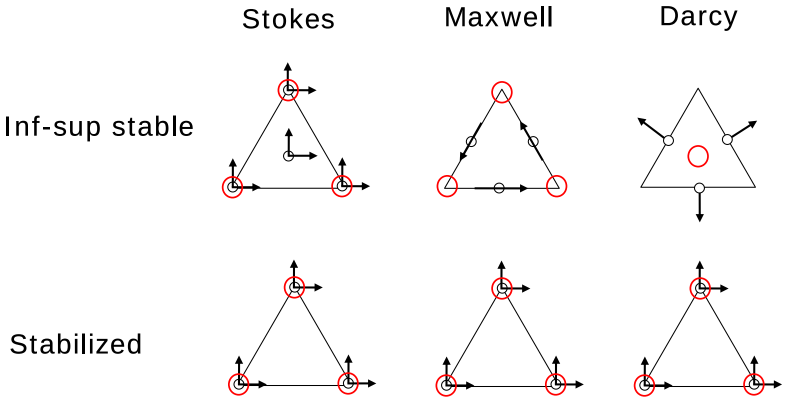
Inf-sup stable pairs

If the Galerkin method is used, inf-sup stable pairs $V_h \times Q_h$ are required. If one can construct finite element spaces such that the following diagram commutes

$$\begin{array}{ccccccc}
\mathbb{R} \to H^1(\Omega) & \xrightarrow{\nabla} & H(\mathbf{curl}) & \xrightarrow{\nabla\times} & H(\mathrm{div}) & \xrightarrow{\nabla\cdot} & L^2(\Omega) \to 0 \\
\downarrow P_h & & \downarrow P_h & & \downarrow P_h & & \downarrow P_h \\
\mathbb{R} \to Q_{h,1} & \xrightarrow{\nabla} & V_{h,\mathrm{Maxwell}} & \xrightarrow{\nabla\times} & V_{h,\mathrm{Darcy}} & \xrightarrow{\nabla\cdot} & Q_{h,2} \to 0
\end{array}$$

for appropriate projections P_h, stability is ensured.

Example of stable spaces



Combined problems

Problem: What if we combine problems?

$$-\nu\Delta\boldsymbol{u} + \lambda\nabla\times\nabla\times\boldsymbol{u} + \sigma\boldsymbol{u} + \nabla p = \boldsymbol{f},$$
$$\nabla\cdot\boldsymbol{u} = 0.$$

Stabilised formulations: any conforming $\boldsymbol{u}$-p interpolation is allowed! No stability problems will be found in the limits:

$$\nu\to 0,\quad \sigma\to 0,\quad \lambda\to 0$$

Objectives

- To develop a functional framework for the Stokes-Darcy problem "well behaved" when $\nu \to 0$ (zero viscosity) and when $\sigma \to 0$ (infinite permeability), with $p \in L^2(\Omega)$.
- To select an appropriate functional framework for the Maxwell problem, with $p \in H^1(\Omega)$.
- To propose finite element methods with
    - Optimal stability
    - Optimal convergence

  for arbitrary conforming approximations of $\boldsymbol{u}$ and p, without the difficulty inherent to inf-sup stable elements for the Stokes, the Maxwell and the Darcy problems.

Key ingredients

The design of the method is based on:

- A two scale decomposition of $\boldsymbol{u}$ and p, within the variational multiscale framework (VMS).
- A proper scaling of the problem, which requires the introduction of a length scale.
- A closed form expression for the subscales based on an approximate Fourier analysis of the problem.

These ingredients will not be elaborated here. The methods proposed will be stated without (heuristic) derivation.

Stokes-Darcy's BVP

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, where we consider the Stokes-Darcy (or Brinkman) problem, which consists in finding a velocity $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and a pressure $p : \Omega \longrightarrow \mathbb{R}$ such that

$$
\begin{aligned}
-\nu\Delta\boldsymbol{u} + \sigma\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\
\nabla\cdot\boldsymbol{u} &= g,
\end{aligned}
$$

For simplicity, as boundary conditions we will consider $\boldsymbol{u} = \boldsymbol{0}$ if $\nu > 0$ and $\boldsymbol{n}\cdot\boldsymbol{u} = 0$ if $\nu = 0$.

Variational form

The variational formulation of the problem consists in finding a velocity-pressure pair $[\boldsymbol{u}, p]$ such that

$$B([\boldsymbol{u}, p], [\boldsymbol{v}, q]) = L([\boldsymbol{v}, q]),$$

for all test functions $[\boldsymbol{v}, q]$, where the bilinear form B and the linear form L are defined by

$$\begin{aligned} B([\boldsymbol{u}, p], [\boldsymbol{v}, q]) &= \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \sigma(\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u}), \\ L([\boldsymbol{v}, q]) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle + \langle g, q \rangle. \end{aligned}$$

Functional setting I

Let us introduce the operator

$$\mathcal{L}\boldsymbol{u} := -\nu\Delta\boldsymbol{u} + \sigma\boldsymbol{u},$$

and the associated graph norm

$$\|\boldsymbol{u}\|_{\mathcal{L}}^2 := \nu\|\nabla\boldsymbol{u}\|^2 + \sigma\|\boldsymbol{u}\|^2.$$

Let $V_{\mathcal{L}}$ be obtained as the closure of $C_0^\infty(\Omega)^d$ with respect to this norm. Its dual space $V'_{\mathcal{L}}$ is endowed with the norm

$$\|\boldsymbol{u}\|_{\mathcal{L}'} := \sup_{\boldsymbol{v}\in V_{\mathcal{L}}} \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{\|\boldsymbol{v}\|_{\mathcal{L}}}.$$

Obviously, $V_{\mathcal{L}} = H_0^1(\Omega)^d$, $V'_{\mathcal{L}} = H^{-1}(\Omega)^d$ if $\nu > 0$ and $V_{\mathcal{L}} = V'_{\mathcal{L}} = L^2(\Omega)^d$ if $\nu = 0$.

Functional setting II

A key ingredient is the introduction of a characteristic length scale L_0, which plays a key role in the Darcy problem. The reason is the need to control both $\boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u}$ to obtain stability in $H(\mathrm{div}, \Omega)$.
Let V be the closure of $C_0^\infty(\Omega)^d$ with respect to the norm $\|\boldsymbol{v}\|_{\mathcal{L}} + \sqrt{\sigma} L_0 \|\nabla \cdot \boldsymbol{v}\|$ and Q the closure of $C^\infty(\Omega)/\mathbb{R}$ with respect to $(\nu + \sigma L_0^2)^{-1/2}\|q\| + \|\nabla q\|_{\mathcal{L}'}$. The pair $V \times Q$ reduces to $H_0^1(\Omega)^d \times L^2(\Omega)/\mathbb{R}$ when $\nu > 0$ and to $H_0(\mathrm{div}, \Omega) \times H^1(\Omega)/\mathbb{R}$ when $\nu = 0$. On $V \times Q$ we define

$$\|\![\boldsymbol{v}, q]\|\!|^2 := \|\boldsymbol{v}\|_{\mathcal{L}}^2 + \sigma L_0^2 \|\nabla \cdot \boldsymbol{v}\|^2 + \frac{1}{\nu + \sigma L_0^2}\|q\|^2 + \|\nabla q\|_{\mathcal{L}'}^2.$$

which is the finest norm in which the problem is well posed.

Main result

Theorem (Stability of the continuous problem)

There exists a constant C such that for all $[\boldsymbol{u}, p] \in V \times Q$ there exists $[\boldsymbol{v}, q] \in V_{\mathcal{L}} \times L^2(\Omega)$ for which

$$B([\boldsymbol{u}, p], [\boldsymbol{v}, q]) \geq C |\!|\!| [\boldsymbol{u}, p] |\!|\!| \, \| [\boldsymbol{v}, q] \|_{V_{\mathcal{L}} \times L^2(\Omega)},$$

1. The working norm is optimal. Observe that

$$|\!|\!| [\boldsymbol{v}, q] |\!|\!|^2 = \nu \| \nabla \boldsymbol{u} \|^2 + \frac{1}{\nu} \| p \|^2 + \frac{1}{\nu} \| \nabla p \|^2_{-1} \qquad \text{when } \sigma = 0,$$

$$|\!|\!| [\boldsymbol{v}, q] |\!|\!|^2 = \sigma \| \boldsymbol{u} \|^2 + \sigma L_0^2 \| \nabla \cdot \boldsymbol{u} \|^2 + \frac{1}{\sigma L_0^2} \| p \|^2 + \frac{1}{\sigma} \| \nabla p \|^2 \quad \text{when } \nu = 0.$$

2. Stability in $|\!|\!| \cdot |\!|\!|$ will be obtained provided the data are regular enough. If the data are less regular, stability for $[\boldsymbol{u}, p]$ can be proved in norms weaker than $|\!|\!| \cdot |\!|\!|$.

Stabilised finite element methods

Let V_h and Q_h be the finite element spaces to approximate the velocity and the pressure, respectively. The two methods to be analyzed can be written as follows: find $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = L_s([\boldsymbol{v}_h, q_h]),$$

for all $[\boldsymbol{v}_h, q_h] \in V_h \times Q_h$.

Algebraic subgrid scale (ASGS) method

The forms B_s and L_s are given by:

$$
\begin{aligned}
B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) &= B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\
&+ \tau_p \sum_K \langle \nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h \rangle_K \\
&+ \tau_u \sum_K \langle -\nu \Delta \boldsymbol{u}_h + \sigma \boldsymbol{u}_h + \nabla p_h, \nu \Delta \boldsymbol{v}_h - \sigma \boldsymbol{v}_h + \nabla q_h \rangle_K \\
&+ \tau_f \sum_E \langle [\![\boldsymbol{n} p_h - \nu \partial_n \boldsymbol{u}_h]\!], [\![\boldsymbol{n} q_h + \nu \partial_n \boldsymbol{v}_h]\!] \rangle_E ,
\end{aligned}
$$

$$
\begin{aligned}
L_s([\boldsymbol{v}_h, q_h]) &= L([\boldsymbol{v}_h, q_h]) \\
&+ \tau_p \sum_K \langle g, \nabla \cdot \boldsymbol{v}_h \rangle_K \\
&+ \tau_u \sum_K \langle \boldsymbol{f}, \nu \Delta \boldsymbol{v}_h - \sigma \boldsymbol{v}_h + \nabla q_h \rangle_K ,
\end{aligned}
$$

Stabilisation parameters

We compute them as

$$\tau_p = c_1 \nu + c_2^p \sigma \ell_p^2,$$
$$\tau_u = (c_1 \nu + c_2^u \sigma \ell_u^2)^{-1} h^2,$$
$$\tau_f = (c_1 \nu + c_2^u \sigma \ell_u^2)^{-1} h,$$

with c_1, c_2^p and c_2^u algorithmic constants.
The length scales ℓ_u and ℓ_p, which can be either taken as L_0, h or $(L_0 h)^{1/2}$, appear when introducing scaling coefficients μ_u and μ_p such that $\mu_u |\boldsymbol{f}|^2 + \mu_p |g|^2$ is dimensionally consistent.
Using the approximate Fourier analysis, the stabilisation parameters are found, now depending on μ_u and μ_p. In turn, these scaling coefficients depend on a length scale of the problem that may be taken as L_0 or h.

Orthogonal sub-grid scale stabilisation (OSGS) method

The bilinear form B_s and the linear form L_s in the OSGS method are given by

$$\begin{aligned} B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) &= B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\ &+\tau_p \sum_K \left\langle P^\perp(\nabla \cdot \boldsymbol{u}_h), P^\perp(\nabla \cdot \boldsymbol{v}_h) \right\rangle_K \\ &+\tau_u \sum_K \left\langle P^\perp(-\nu\Delta \boldsymbol{u}_h + \nabla p_h), P^\perp(\nu\Delta \boldsymbol{v}_h + \nabla q_h) \right\rangle_K \\ &+\tau_f \sum_E \langle [\![\boldsymbol{n} p_h - \nu \partial_n \boldsymbol{u}_h]\!], [\![\boldsymbol{n} q_h + \nu \partial_n \boldsymbol{v}_h]\!] \rangle_E, \\ L_s([\boldsymbol{v}_h, q_h]) &= L([\boldsymbol{v}_h, q_h]). \end{aligned}$$

The stabilisation parameters are the same as for the ASGS method.

Working norm and error function

Let us define the mesh dependent norm

$$
\begin{aligned}
|||[\boldsymbol{v}_h, q_h]|||_h^2 &= \|\boldsymbol{v}_h\|_{\mathcal{L}}^2 + \sigma \ell_p^2 \|\nabla \cdot \boldsymbol{v}_h\|^2 + \frac{1}{\nu + \sigma L_0^2} \|q_h\|^2 \\
&+ \frac{h^2}{\nu + \sigma \ell_u^2} \sum_K \|\nabla q_h\|_K^2 + \frac{h}{\nu + \sigma \ell_u^2} \sum_E \| [\![\boldsymbol{n} q_h]\!] \|_E^2
\end{aligned}
$$

We also define

$$
\begin{aligned}
E(h)^2 &= (\nu + \sigma \ell_p^2)(h^{-2} \varepsilon_0^2(\boldsymbol{u}) + \varepsilon_1^2(\boldsymbol{u})) + \sigma \varepsilon_0^2(\boldsymbol{u}) \\
&+ \frac{h^2}{\nu + \sigma \ell_u^2} (h^{-2} \varepsilon_0^2(p) + \varepsilon_1^2(p)).
\end{aligned}
$$

It can be proved that these are the norm and error function of the methods to be analyzed ($\varepsilon_i(v)$ is the interpolation error of function v in the norm of $H^i(\Omega)$).

Analysis I

Theorem (Stability)

Suppose that the constants c_1 and c_2^u are large enough. Then, there exists a constant C such that

$$\forall[\boldsymbol{u}_h, p_h]\, \exists[\boldsymbol{v}_h, q_h] \mid$$
$$B_s([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \geq C |\!|\!| [\boldsymbol{u}_h, p_h] |\!|\!|_h |\!|\!| [\boldsymbol{v}_h, q_h] |\!|\!|_h,$$

Let us compare the working norms of the continuous and the discrete problems, for simplicity in the case of continuous pressure interpolations:

$$|\!|\!| [\boldsymbol{v}, q] |\!|\!|^2 = \|\boldsymbol{v}\|_{\mathcal{L}}^2 + \sigma L_0^2 \|\nabla \cdot \boldsymbol{v}\|^2 + \frac{1}{\nu + \sigma L_0^2} \|q\|^2 + \|\nabla q\|_{\mathcal{L}'}^2,$$

$$|\!|\!| [\boldsymbol{v}_h, q_h] |\!|\!|_h^2 = \|\boldsymbol{v}_h\|_{\mathcal{L}}^2 + \sigma \ell_p^2 \|\nabla \cdot \boldsymbol{v}_h\|^2 + \frac{1}{\nu + \sigma L_0^2} \|q_h\|^2 + \frac{h^2}{\nu + \sigma \ell_u^2} \|\nabla q_h\|^2.$$

Analysis II

Theorem (Convergence)

Let $[\boldsymbol{u}, p]$ be the solution of the continuous problem and $[\boldsymbol{u}_h, p_h]$ the solution of the discrete one. Suppose that $\ell_p \geq \ell_u$ and the assumptions of the previous theorem hold. Then

$$|||[\boldsymbol{u} - \boldsymbol{u}_h, p - p_h]|||_h \lesssim E(h)$$

Accuracy of the stabilised formulations for Darcy's problem

Method $\ell_p, \ell_u =$	A h, h	B ${L_0}^{1/2}h^{1/2}, {L_0}^{1/2}h^{1/2}$	C L_0, L_0
$\Vert \boldsymbol{e}_u \Vert$ Original	$h^{k+1} + h^l$ Suboptimal	$h^{k+1/2} + h^{l+1/2}$ Quasi-optimal	$h^k + h^{l+1}$ Suboptimal
$\Vert \boldsymbol{e}_u \Vert$ Via duality	$h^{k+1} + h^l$ Suboptimal	$h^{k+1} + h^{l+1}$ Optimal	$h^k + h^{l+1}$ Suboptimal
$\Vert e_p \Vert$ Original	$h^{k+1} + h^l$ Suboptimal	$h^{k+1/2} + h^{l+1/2}$ Quasi-optimal	$h^k + h^{l+1}$ Suboptimal
$\Vert e_p \Vert$ Via duality	$h^{k+2} + h^{l+1}$ Optimal	$h^{k+1} + h^{l+1}$ Optimal	$h^k + h^{l+1}$ Suboptimal
$\Vert \nabla \cdot \boldsymbol{e}_u \Vert$	$h^k + h^{l-1}$ Suboptimal	$h^k + h^l$ Optimal	$h^k + h^{l+1}$ Optimal
$\Vert \nabla e_p \Vert$	$h^{k+1} + h^l$ Optimal	$h^k + h^l$ Optimal	$h^{k-1} + h^l$ Suboptimal
$k,\ l$ Optimal	$k+1=l$	$k=l$	$k=l+1$

k, l: Velocity and pressure interpolation order.

Maxwell's BVP

Original problem: minimise the functional

$$\mathcal{E}(\boldsymbol{v}) = \int_\Omega \left(\frac{\lambda}{2} |\nabla \times \boldsymbol{v}|^2 - 2\boldsymbol{v} \cdot \boldsymbol{f} \right),$$

with the constraint $\nabla \cdot \boldsymbol{v} = 0$ and $\boldsymbol{n} \times \boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega$, with $\nabla \cdot \boldsymbol{f} = 0$.
Augmented BVP: find $[\boldsymbol{u}, p]$ such that (sign of p changed)

$$\begin{aligned}
\lambda \nabla \times (\nabla \times \boldsymbol{u}) - \nabla p = \boldsymbol{f}, &\quad \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u} = 0, &\quad \text{in } \Omega, \\
\boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{0}, &\quad \text{on } \partial\Omega, \\
p = 0, &\quad \text{on } \partial\Omega.
\end{aligned}$$

curl and curl-div formulations

curl formulation: find $\boldsymbol{u} \in H_0(\mathbf{curl})$ and $p \in H_0^1(\Omega)$ such that

$$(\lambda \nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) - (\nabla p, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in H_0(\mathbf{curl}),$$
$$(\nabla q, \boldsymbol{u}) = 0, \qquad \forall q \in H_0^1(\Omega).$$

curl-div formulation I: find $\boldsymbol{u} \in H_0(\mathbf{curl}) \cap H(\mathrm{div})$ and $p \in L^2(\Omega)/\mathbb{R}$ such that

$$(\lambda \nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + (p, \nabla \cdot \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in H_0(\mathbf{curl}) \cap H(\mathrm{div}),$$
$$-(q, \nabla \cdot \boldsymbol{u}) = 0, \qquad \forall q \in L^2(\Omega).$$

In any case, $p = 0$.
curl-div formulation II:

$$(\lambda \nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + (\lambda \nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in H_0(\mathbf{curl}) \cap H(\mathrm{div}).$$

Notation

Consider:

$$a(\boldsymbol{u},\boldsymbol{v}) = (\lambda\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{u})\,,\quad b(\boldsymbol{v},p) = -(\nabla p,\boldsymbol{v})\,,$$
$$B(\boldsymbol{u},p;\boldsymbol{v},q) = a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) - b(\boldsymbol{u},q).$$

Let $V = H_0(\mathbf{curl})$ and $Q = H_0^1(\Omega)/\mathbb{R}$, with norms:

$$\|\boldsymbol{v}\|_V := \|\boldsymbol{v}\|_{H(\mathbf{curl})} = \frac{1}{L_0}\|\boldsymbol{v}\| + \|\nabla\times\boldsymbol{v}\|,$$
$$\|q\|_Q := \|q\|_{H_0^1(\Omega)} = \frac{1}{L_0}\|q\| + \|\nabla q\|,$$

where $L_0 = L_0(\Omega)$ is a length scale. The norm associated to the product space $V\times Q$ is denoted by

$$|||\boldsymbol{v},q||| = \lambda^{\frac{1}{2}}\|\boldsymbol{v}\|_V + L_0\lambda^{-\frac{1}{2}}\|q\|_Q.$$

Stability

Theorem

The following inf-sup condition is satisfied,

$$\inf_{[\boldsymbol{u},p]\in V\times Q} \sup_{[\boldsymbol{v},q]\in V\times Q} \frac{B(\boldsymbol{u},p;\boldsymbol{v},q)}{|||\boldsymbol{u},p|||\,|||\boldsymbol{v},q|||} \geq \beta > 0.$$

- This guarantees well-posedness of the continuous problem.
- The solution satisfies $\nabla \cdot \boldsymbol{u} = 0$, but there is no control on $\|\nabla \cdot \boldsymbol{u}\|$.

Galerkin approximation: curl-div formulation

Nodal finite element spaces:

$$\mathcal{N}_k(\Omega) = \left\{ \boldsymbol{v}_h \in \mathcal{C}^0(\Omega) \text{ such that } \boldsymbol{v}_h|_K \in \mathcal{P}_k(K)\ \forall K \in \mathcal{T}_h \right\}.$$

Discrete curl-div formulation: find $\boldsymbol{u}_h \in X_h$ such that

$$(\lambda \nabla \times \boldsymbol{u}_h, \nabla \times \boldsymbol{v}_h) + (\lambda \nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in X_h,$$

where X_h is a H^1-conforming finite element space.

The corner paradox

Lemma

If Ω is not convex, $V \cap H^1(\Omega)^d$ is a closed proper subspace of $V \cap H(\mathrm{div})$.

Corollary

If Ω is not convex

$$\lim_{h \to 0} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{V \cap H(\mathrm{div})} \neq 0,$$

in general.

Approximability problem?

Key fact:

$$\text{If } \boldsymbol{u}_h \in H^1(\Omega)^d \;\Rightarrow\; \|\nabla \boldsymbol{u}_h\| \lesssim \|\nabla \times \boldsymbol{u}_h\| + \|\nabla \cdot \boldsymbol{u}_h\| \qquad (7)$$

Wrong conclusion:

H^1 conforming FE spaces cannot approximate V.

Correct conclusion:

If $\|\nabla \cdot \boldsymbol{u}_h\|$ is uniformly bounded, H^1 conforming FE spaces cannot approximate V.

Discrete problem: curl formulation

Mixed formulation:

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) &= (\boldsymbol{f}, \boldsymbol{v}_h)\,, && \forall \boldsymbol{v}_h \in V_h, \\ -b(\boldsymbol{u}_h, q_h) &= 0, && \forall q_h \in Q_h. \end{aligned}$$

Inf-sup condition:

$$\inf_{[\boldsymbol{u}_h, p_h] \in V_h \times Q_h} \sup_{[\boldsymbol{v}_h, q_h] \in V_h \times Q_h} \frac{B(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h)}{|||\boldsymbol{u}_h, p_h|||\,|||\boldsymbol{v}_h, q_h|||} \geq \beta_d > 0.$$

Compatible pairs:

V_h constructed using Nédélec's elements.

Q_h constructed using a nodal interpolation.

A novel augmented formulation

Equivalent problem:

$$\lambda \nabla \times \nabla \times \boldsymbol{u} - \nabla p = \boldsymbol{f},$$
$$-\nabla \cdot \boldsymbol{u} - \frac{{L_0}^2}{\lambda} \Delta p = 0,$$

in Ω, satisfying $\boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{0}$ and $p = 0$ on $\partial\Omega$.
Weak form:

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in V,$$
$$-b(\boldsymbol{u}, q) + s_p(p, q) = 0, \qquad \forall q \in Q,$$

where

$$s_p(p, q) = \frac{{L_0}^2}{\lambda} \int_\Omega \nabla p \cdot \nabla q.$$

Stabilised finite element approximation

New formulation:

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) + s_u(\boldsymbol{u}_h, \boldsymbol{v}_h) &= (\boldsymbol{f}, \boldsymbol{v}_h)\,, & \forall \boldsymbol{v}_h \in V_h, \\ -b(\boldsymbol{u}_h, q_h) + s_p(p_h, q_h) &= 0, & \forall q_h \in Q_h, \end{aligned}$$

where the stabilisation term reads

$$s_u(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{K \in \mathcal{T}_h} c_u \lambda \int_K \frac{h_K^2}{{L_0}^2} \nabla \cdot \boldsymbol{u}_h \nabla \cdot \boldsymbol{v}_h$$

This formulation can also be obtained using the VMS framework with an appropriate scaling of the equations for the subscales and a closed form approximation based on a Fourier analysis.

Stability I

Bilinear form:

$$B_s(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) = B(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + s_u(\boldsymbol{u}_h, \boldsymbol{v}_h) + s_p(p_h, q_h)$$

Mesh-dependent norm:

$$|||\boldsymbol{v}_h, q_h|||_h = \lambda^{\frac{1}{2}} \|\nabla \times \boldsymbol{v}_h\| + \lambda^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}_h} \frac{h_K^2}{{L_0}^2} \|\nabla \cdot \boldsymbol{u}_h\|_K^2 \right)^{\frac{1}{2}} + \frac{L_0}{\lambda^{\frac{1}{2}}} \|\nabla p_h\|.$$

Lemma (Stability in the mesh dependent norm)

The bilinear form $B_s : V_h \times Q_h \times V_h \times Q_h \to \mathbb{R}$ is coercive with respect to the mesh-dependent norm.

Stability II

Lemma (Norm equivalence)

The solution $[\boldsymbol{w}_h, \alpha_h] \in V_h \times Q_h$ *of the discrete problem*

$$B_s(\boldsymbol{w}_h, \alpha_h; \boldsymbol{v}_h, q_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle + \langle g, q_h \rangle, \quad \forall (\boldsymbol{v}_h, q_h) \in V_h \times Q_h,$$

for $\boldsymbol{f} \in V'$ *and* $g \in Q'$*, satisfies:*

$$|||\boldsymbol{w}_h, \alpha_h|||_h \lesssim |||\boldsymbol{w}_h, \alpha_h||| \lesssim |||\boldsymbol{w}_h, \alpha_h|||_h + \|g\|_{Q'}.$$

Corollary (Natural stability)

The solution $[\boldsymbol{u}_h, p_h]$ *of the problem satisfies*

$$|||\boldsymbol{u}_h, p_h||| \lesssim \|\boldsymbol{f}\|.$$

Convergence I

Corollary (Natural continuity)

The stabilised bilinear form $B_s : V_h \times Q_h \times V_h \times Q_h \to \mathbb{R}$ is continuous with respect to the norm $|||\cdot|||$.

Error function:

$$E_h(\boldsymbol{u}) := \inf_{[\boldsymbol{w}_h, r_h] \in V_h \times Q_h} \Bigg[|||\boldsymbol{u} - \boldsymbol{w}_h, p - r_h||| + \lambda^{\frac{1}{2}} \left(\sum_K \frac{h_K}{{L_0}^2} \|\boldsymbol{u} - \boldsymbol{w}_h\|^2_{L^2(\partial K)} \right)^{\frac{1}{2}} \Bigg].$$

Convergence II

Theorem (Convergence)

The solution $[\boldsymbol{u}_h, p_h]$ *of the discrete problem satisfies*

$$|||\boldsymbol{u}_h - \boldsymbol{u}, p_h - p||| \lesssim E_h(\boldsymbol{u}).$$

Interpolation estimates:

$$\inf_{\boldsymbol{w}_h \in V_h} \|\boldsymbol{v} - \boldsymbol{w}_h\|_{H^s(\omega)} \lesssim h^{t-s} \|\boldsymbol{v}\|_{H^t(\omega)}, \qquad 0 \leq s \leq t \leq k+1,$$

$$\inf_{r_h \in Q_h} \|q - r_h\|_{H^s(\omega)} \lesssim h^{t-s} \|q\|_{H^t(\omega)}, \qquad 0 \leq s \leq t \leq l+1,$$

for any bounded set $\omega \subset \Omega$.

Convergence III

Corollary (Convergence to smooth solutions)

If $\boldsymbol{u} \in H^r(\Omega)^d$, with $r \geq 1$, the solution $[\boldsymbol{u}_h, p_h]$ satisfies:

$$|||\boldsymbol{u} - \boldsymbol{u}_h, p - p_h||| \lesssim \lambda^{\frac{1}{2}} h^{t-1} \|\boldsymbol{u}\|_{H^t(\Omega)}, \quad t := \min\{r, k+1\}.$$

Lemma (Decomposition of singular solutions)

The solution $\boldsymbol{u} \in V \cap H(\mathrm{div})$ of the problem can be decomposed into a regular part and a singular part as follows:

$$\boldsymbol{u} = \boldsymbol{u}_0 + \nabla\varphi,$$

where $\boldsymbol{u}_0 \in H^{1+r}(\Omega)^d \cap H_0(\mathbf{curl})$, $\varphi \in H_0^1(\Omega) \cap H^{1+r}(\Omega)$ for some real number $r > \frac{1}{2}$.

Convergence IV

Assumption on the finite element mesh: There exists a finite element space G_h defined over $\mathcal{T}_h$ such that, for any $\phi_h \in G_h$, $\nabla\phi_h \in V_h$. Furthermore, this space satisfies

$$\inf_{\phi_h \in G_h} \|\phi - \phi_h\|_{H^s(\omega)} \lesssim h^{t-s}\|\phi\|_{H^t(\omega)}$$

for $\phi \in H^t(\omega)$ and $0 \leq s \leq t \leq 1 + k$.

Corollary (Convergence to singular solutions)

Under the previous assumption, the solution $[\boldsymbol{u}_h, p_h]$ *of the discrete problem satisfies*

$$|||\boldsymbol{u} - \boldsymbol{u}_h, p - p_h||| \lesssim \lambda^{\frac{1}{2}} h^t \|\boldsymbol{u}_0\|_{H^{1+t}(\Omega)} + \frac{\lambda^{\frac{1}{2}}}{{L_0}^{1-\epsilon}} h^{t-\epsilon}\|\varphi\|_{H^{1+t}(\Omega)},$$

for any $\epsilon \in]0, t - 1/2[$ *and for* $t = \min\{r, k\}$.

Darcy flow I

We consider a problem in the unit square with analytical solution:

$$\boldsymbol{u} = (-2\pi \cos(2\pi x) \sin(2\pi y), -2\pi \sin(2\pi x) \cos(2\pi y))$$
$$p = \sin(2\pi x) \sin(2\pi y)$$

Method $\ell_p, \ell_u =$	A h, h	B $L_0{}^{1/2} h^{1/2}, L_0{}^{1/2} h^{1/2}$	C L_0, L_0
$\Vert \boldsymbol{e}_u \Vert$	1.92 (1)	2.05 (2)	1.97 (1)
$\Vert e_p \Vert$	1.93 (2)	2.20 (2)	2.04 (1)
$\Vert \nabla \cdot \boldsymbol{e}_u \Vert$	1.30 (-)	1.43 (1)	1.43 (1)
$\Vert \nabla e_p \Vert$	1.99 (1)	1.70 (1)	0.58 (-)

Table: Experimental convergence rates for the stabilised method according to the choice of the length scale in the stabilisation parameters when $\nu = 0$. The P1-P1 pair.

Darcy flow II

Method $\ell_p, \ell_u =$	A h, h	B ${L_0}^{1/2}h^{1/2}, {L_0}^{1/2}h^{1/2}$	C L_0, L_0
$\|\|\boldsymbol{e}_u\|\|$	-0.03 (-)	0.86 (1)	1.90 (1)
$\|\|e_p\|\|$	-0.04 (-)	0.96 (1)	1.84 (1)
$\|\|\nabla \cdot \boldsymbol{e}_u\|\|$	-0.42 (-)	0.55 (-)	1.56 (1)

Table: Experimental convergence rates for the stabilised method according to the choice of the length scale in the stabilisation parameters when $\nu = 0$. Piecewise constant pressures.

Stokes flow I

We consider a problem in the unit square with analytical solution:

$$\boldsymbol{u} = (2\pi \sin(2\pi x)\cos(2\pi y), -2\pi \cos(2\pi x)\cos(2\pi y))$$
$$p = \cos(2\pi x)\cos(2\pi y) - 1$$

Method	ASGS	OSGS
$\|\|\boldsymbol{e}_u\|\|$	1.91 (2)	2.00 (2)
$\|\|e_p\|\|$	1.67 (1)	1.99 (1)
$\|\|\nabla \cdot \boldsymbol{e}_u\|\|$	1.63 (1)	1.48 (1)
$\|\|\nabla e_p\|\|$	0.55 (-)	0.58 (-)

Table: Experimental convergence rates for the OSGS and ASGS methods when $\sigma = 0$. The P1-P1 pair.

Stokes flow II

Method	ASGS	OSGS
$\|\|\boldsymbol{e}_u\|\|$	1.98 (2)	1.98 (2)
$\|\|e_p\|\|$	1.45 (1)	1.42 (1)
$\|\|\nabla \cdot \boldsymbol{e}_u\|\|$	1.43 (1)	1.31 (1)

Table: Experimental convergence rates for the OSGS and ASGS methods when $\sigma = 0$. Piecewise constant pressures.

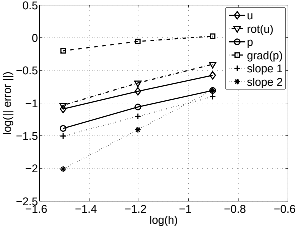
Maxwell's problem I

We take the datum $\boldsymbol{f}$ such that the solution in polar coordinates (r, θ) is:

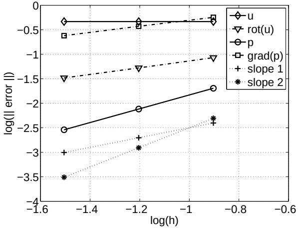
$$\boldsymbol{u} = \nabla \left(r^{\frac{2n}{3}} \sin \frac{2n\theta}{3} \right)$$

in the nonconvex domain $\Omega \equiv [-1, 1]^2 \setminus [0, 1]^2$. We have that $\boldsymbol{u} \in H^{\frac{2n}{3}-\epsilon}(\Omega)$, for any $\epsilon > 0$.

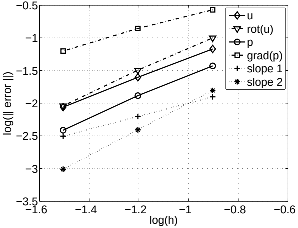
Maxwell's problem II



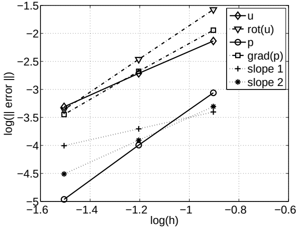
(a) $n = 1$



(b) $n = 1$ without div-div term



(c) $n = 2$



(d) $n = 4$

Maxwell's problem III

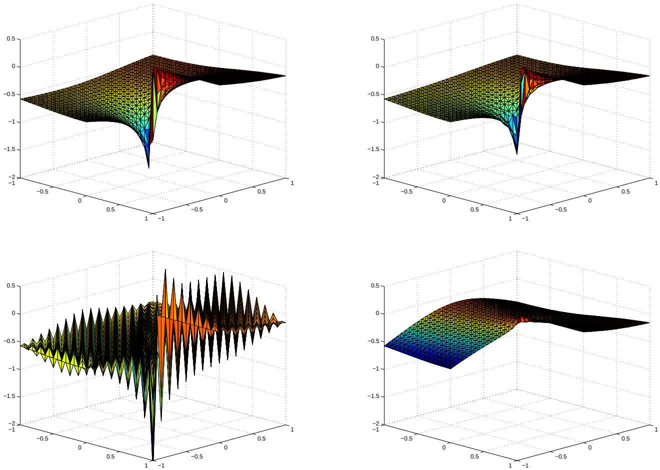
Instead of stabilising the (augmented) curl-formulation, one could stabilise the mixed curl-div formulation, leading to:

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) + (c_u \lambda \nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h),$$

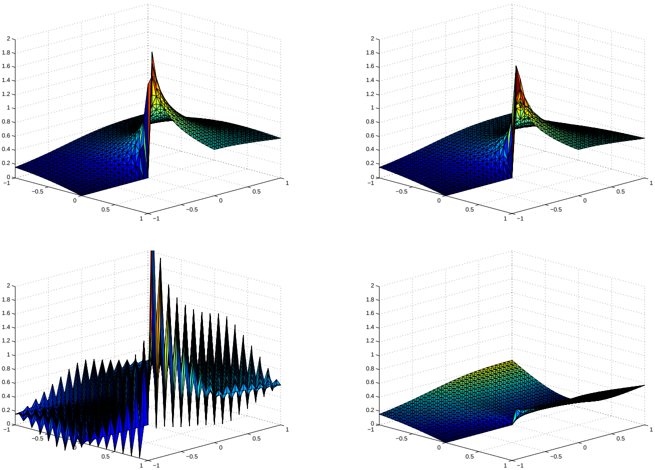
$$-b(\boldsymbol{u}_h, q_h) + \sum_{K \in \mathcal{T}_h} \int_K \frac{h_K^2}{\lambda} \nabla p_h \cdot \nabla q_h = 0,$$

This formulation should not be able to approximate solutions $\boldsymbol{u} \notin H^1(\Omega)^d$.

Maxwell's problem IV: u_x-velocity



Maxwell's problem V: u_y-velocity



Conclusions I: Stokes-Darcy's problem

- A unified framework for the Stokes-Darcy problem.
- Two families of stabilised finite element methods which are optimally stable and convergent.
- A design of the stabilisation parameters which, according to the choice of a length scale, yield methods with different properties in the case of Darcy's problem:
  - Method A: Classical stabilised finite element method. Optimal only when the problem is viewed as a mixed Poisson's problem.
  - Method B: Optimal for $k = l$.
  - Method C: Optimal if $k = l + 1$, which would correspond to inf-sup stable elements for the Stokes problem.
- In any case, arbitrary $\boldsymbol{u}$-p interpolations can be used (conforming, in the formulation presented here).

Conclusions II: Maxwell's problem

Main ingredients of the numerical method:

- Use of a mixed formulation to avoid spurious control on $\|\nabla \cdot \boldsymbol{u}_h\|$.
- Augmented formulation to control $\|\nabla p_h\|$.
- stabilising terms to control $h\|\nabla \cdot \boldsymbol{u}_h\|$ and thus the whole $H(\mathbf{curl})$-norm.

Main properties:

- Stability in natural norms.
- Optimal convergence for smooth solutions.
- Excellent convergence to a singular solution in a test case.
- Arbitrary (conforming) $\boldsymbol{u}$-p interpolations can be used (including nodal interpolations).

Conclusions III: general remarks

- The difficulty of a unified framework for the Stokes-Maxwell-Darcy problem relies on the different functional setting for p ($p \in L^2(\Omega)$ for the Stokes and the Darcy problems, $p \in H^1(\Omega)$ for the Maxwell problem).
- In all cases, the stabilised formulations can be based on the VMS formulation with:
  - A proper scaling of the problem, which requires the introduction of L_0.
  - A closed form expression for the subscales based on an approximate Fourier analysis.
- The methods analyzed fit in the objective of making the numerical formulation independent of the finite element interpolation.