HYBRIDISABLE DISCONTINUOS GALERKIN METHODS

PART 1: FROM DG TO HDG FORMULATIONS

Matteo Giacomini

XXII Spanish-French School Jacques-Louis Lions
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Thanks to…

My co-authors:

Antonio Huerta
(UPC-CIMNE)

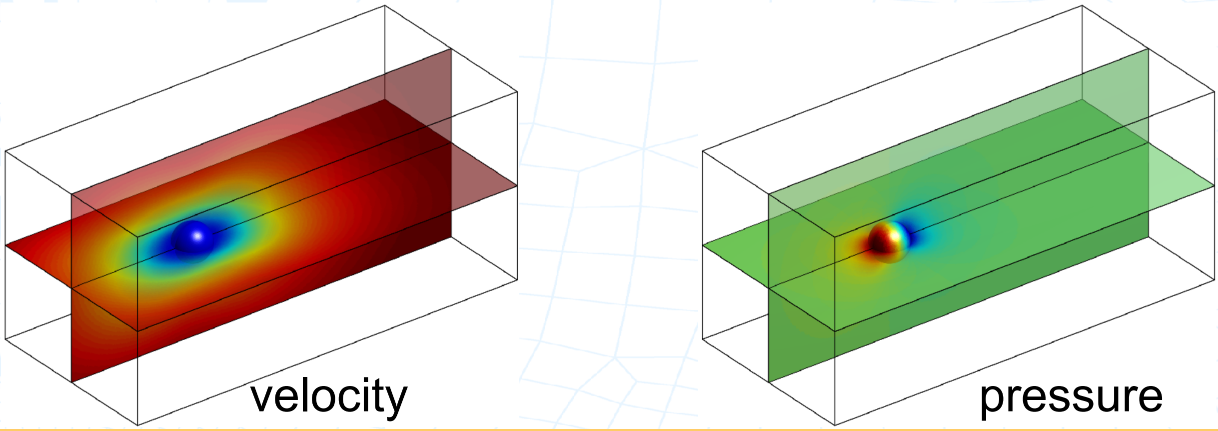
Rubén Sevilla
(Swansea University)

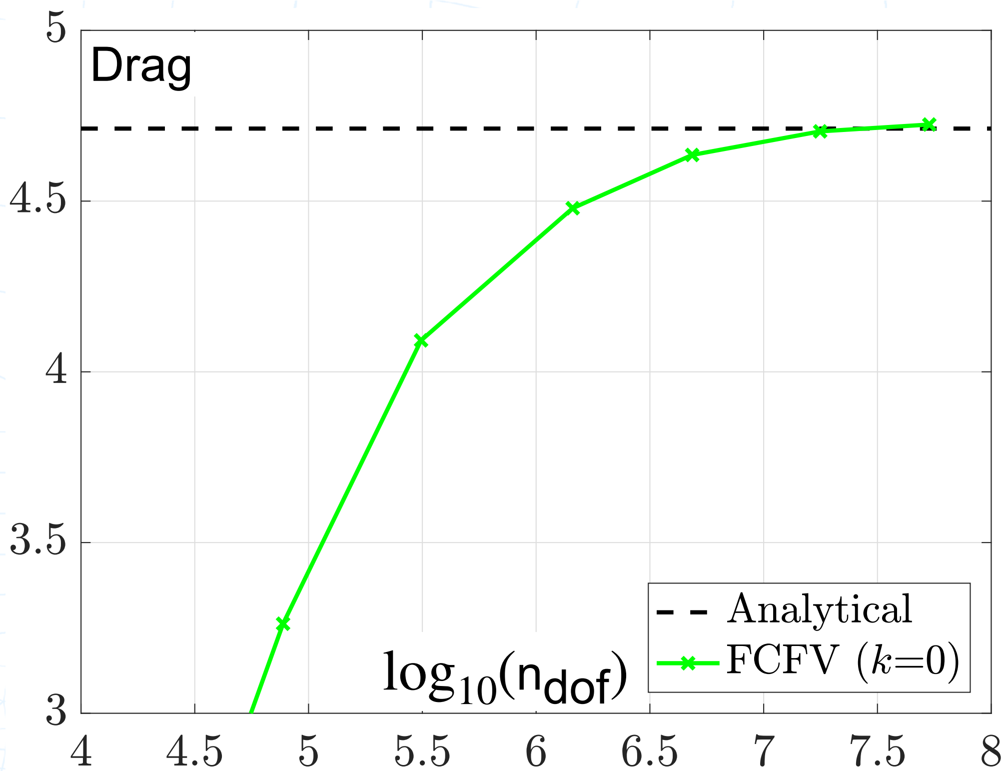
Former PhD students:

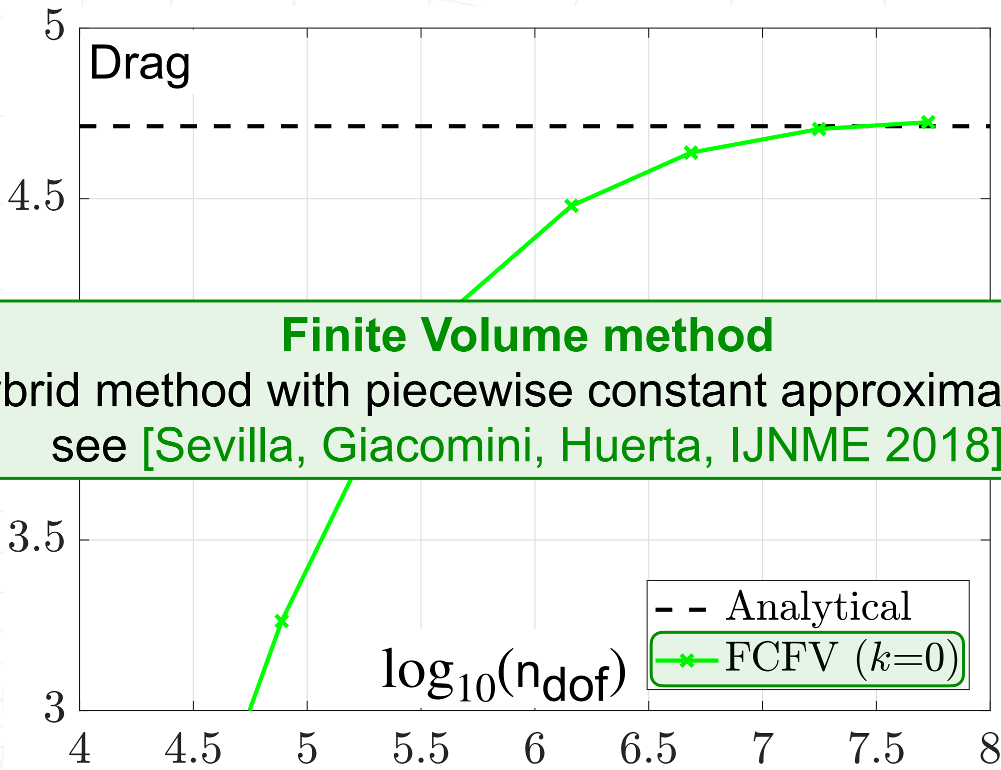
Alexandros Karkoulias (2020)
Jordi Vila-Pérez (2021)
Davide Cortellessa (2024)
Luan M. Vieira (2024)

Motivation

- **Industry requires** both **Robust** and **High-Fidelity** solvers for CFD
  Robust solvers are often linked to *Finite Volumes*, while
  High-Fidelity solvers rely on *high-order approximations*.
- A simple example with analytical solution: Stokes flow past a sphere







Finite Volume method
(hybrid method with piecewise constant approximations)
see [Sevilla, Giacomini, Huerta, IJNME 2018]

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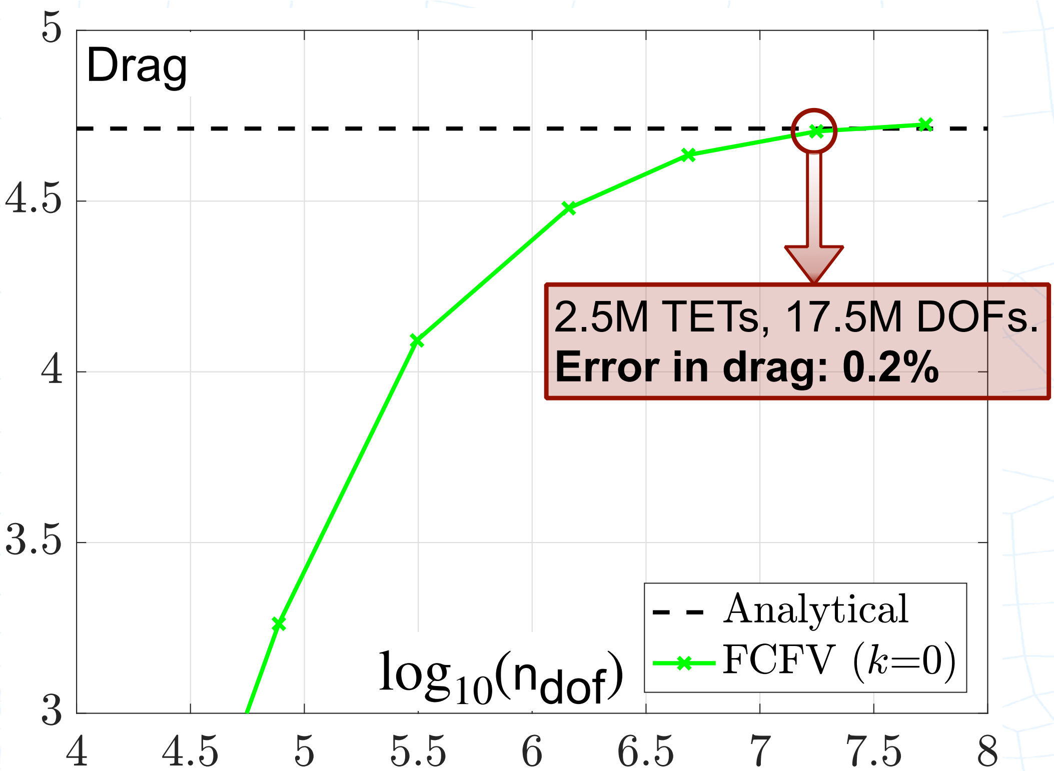

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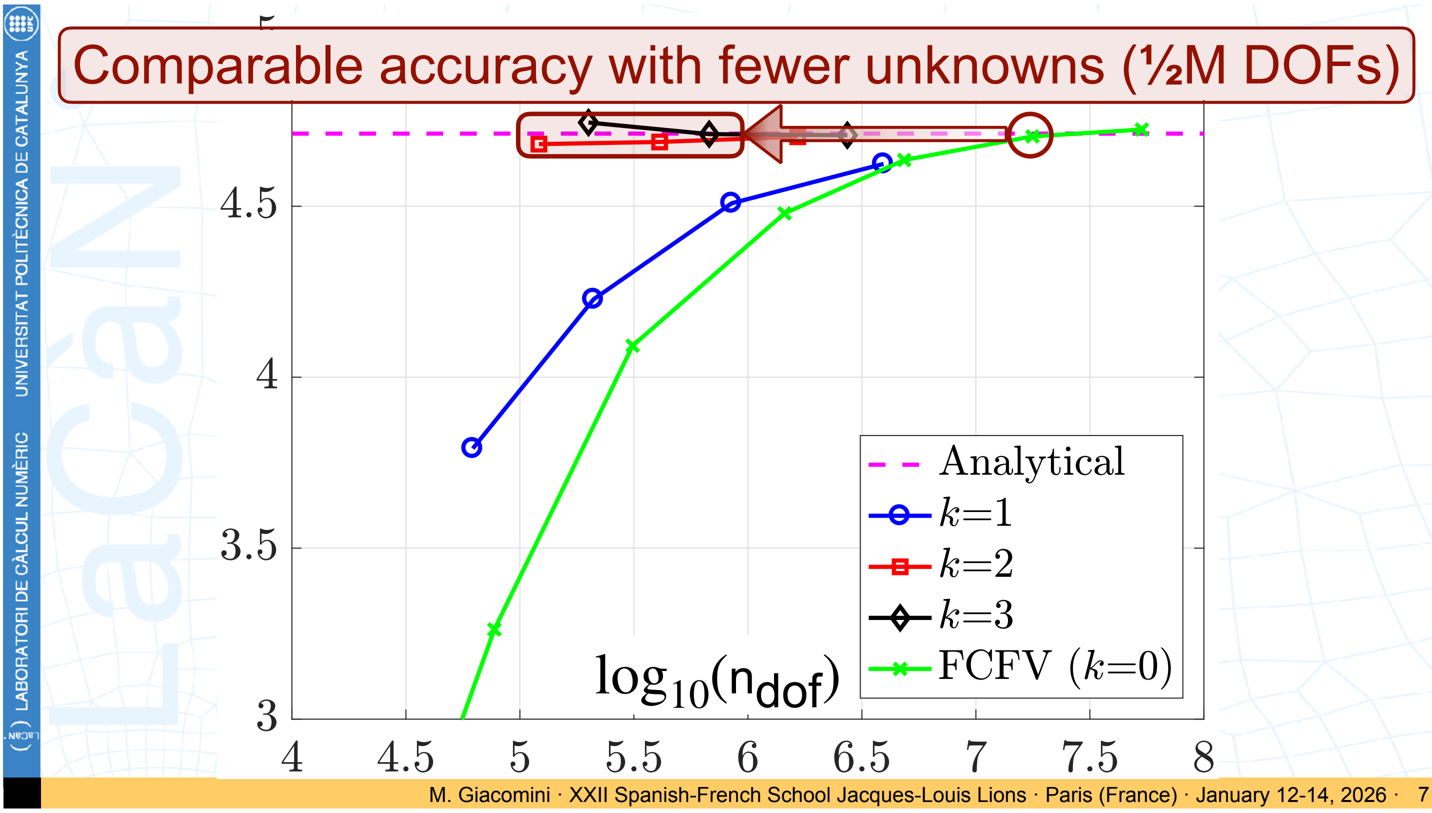
Comparable accuracy with fewer unknowns (½M DOFs)
4.5
4
3.5
3
4
4.5
5
5.5
6
6.5
7
7.5
8
$\log_{10}(n_{dof})$
Analytical
k=1
k=2
k=3
FCFV (k=0)
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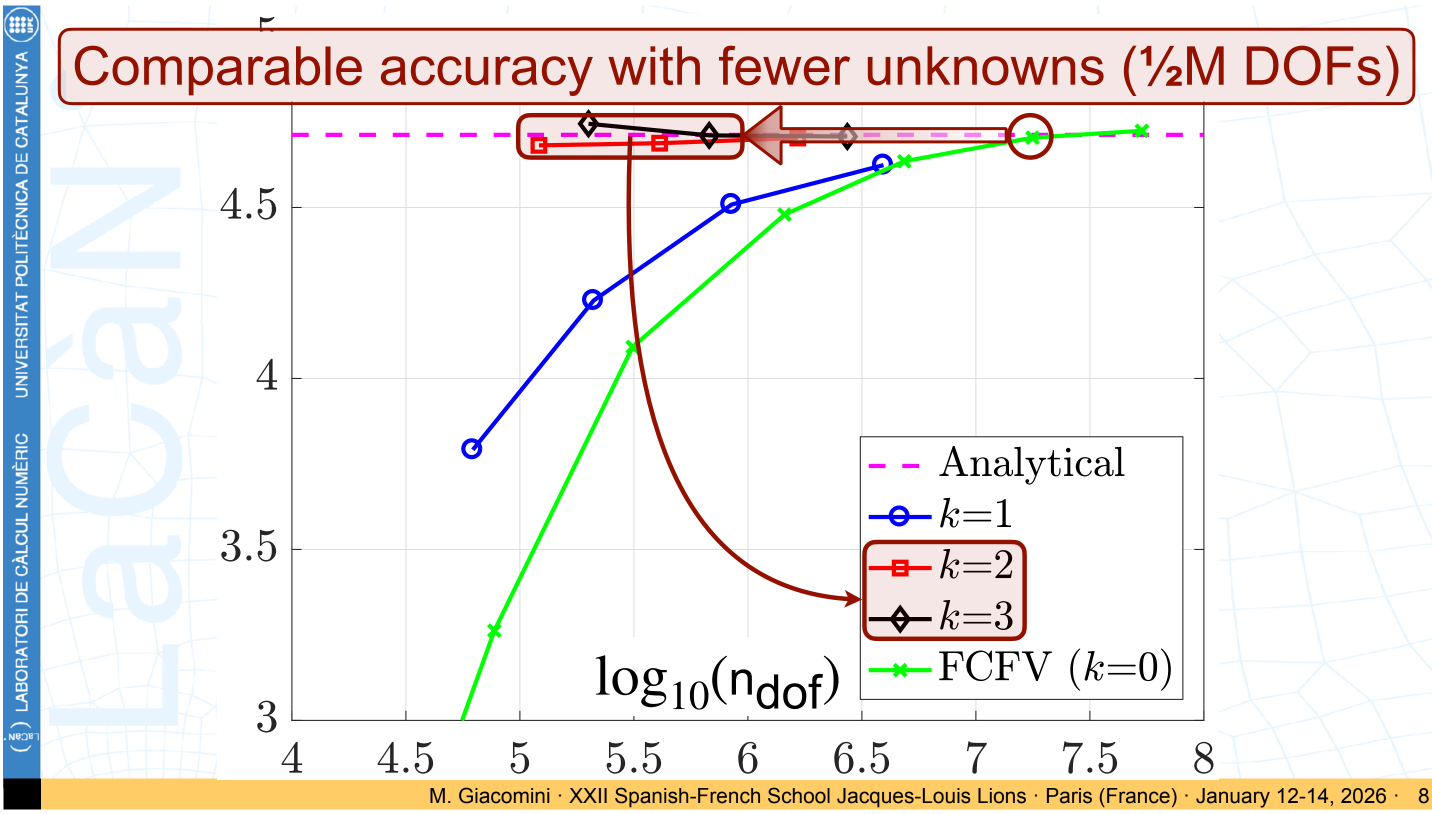
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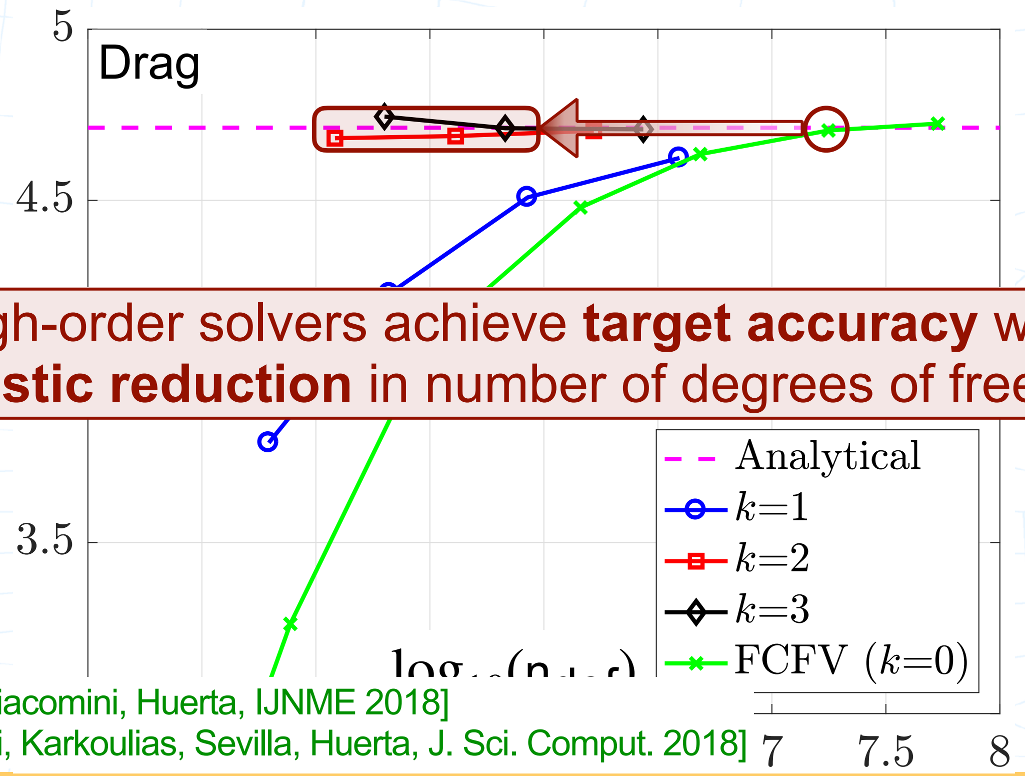

High-order solvers achieve **target accuracy** with a **drastic reduction** in number of degrees of freedom

[Sevilla, Giacomini, Huerta, IJNME 2018]
[Giacomini, Karkoulias, Sevilla, Huerta, J. Sci. Comput. 2018]

Finite Volumes

- ✓ Widely used in industry, both **open-source** and **commercial** software.
- ✓ **Robust** and efficient for tackling large-scale problems.
- ✓ **Mesh generation** handled by a **large number of codes**.

High-Fidelity

- ✓ Efficient in **delivering target accuracy**.
- ✓ **Resolves scales** in turbulent flows and aeroacoustic problems.
- ✓ Handles transient phenomena with **low diffusion** & **dispersion** error.

Finite Volumes

- ✓ Widely used in industry, both **open-source** and **commercial** software.
- ✓ **Robust** and efficient for tackling large-scale problems.
- ✓ **Mesh generation** handled by a **large number of codes**.
- ✗ **Many degrees of freedom** for target accuracy.
- ✗ **Excessive mesh sensitivity**.
- ✗ Issues in **incompressible limit**.
- ✗ **Cannot resolve fine scales**.
- ✗ Poor for transient because of **high dispersion and diffusion**.

High-Fidelity

- ✓ Efficient in **delivering target accuracy**.
- ✓ **Resolves scales** in turbulent flows and aeroacoustic problems.
- ✓ Handles transient phenomena with **low diffusion** & **dispersion** error.
- ✗ **Predominantly academic**, not yet mature for industrial requirements.
- ✗ Needs **additional techniques** to ensure monotonicity **in shocks**.
- ✗ **Difficult** high-order **mesh generation** of real configurations.

Outline

1. Review of DG methods: IPM & LDG
2. HDG formulation
3. Accuracy and computational cost

4. HDGlab: implementation aspects
5. HDG for mechanical problems
6. HDG with convection stabilisation via Riemann solvers
7. HDG with exact geometry and degree adaptivity
8. From HDG (high-order) to FCFV (low-order)
9. Some ongoing research lines

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A brief review of Discontinuous Galerkin methods

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Definitions

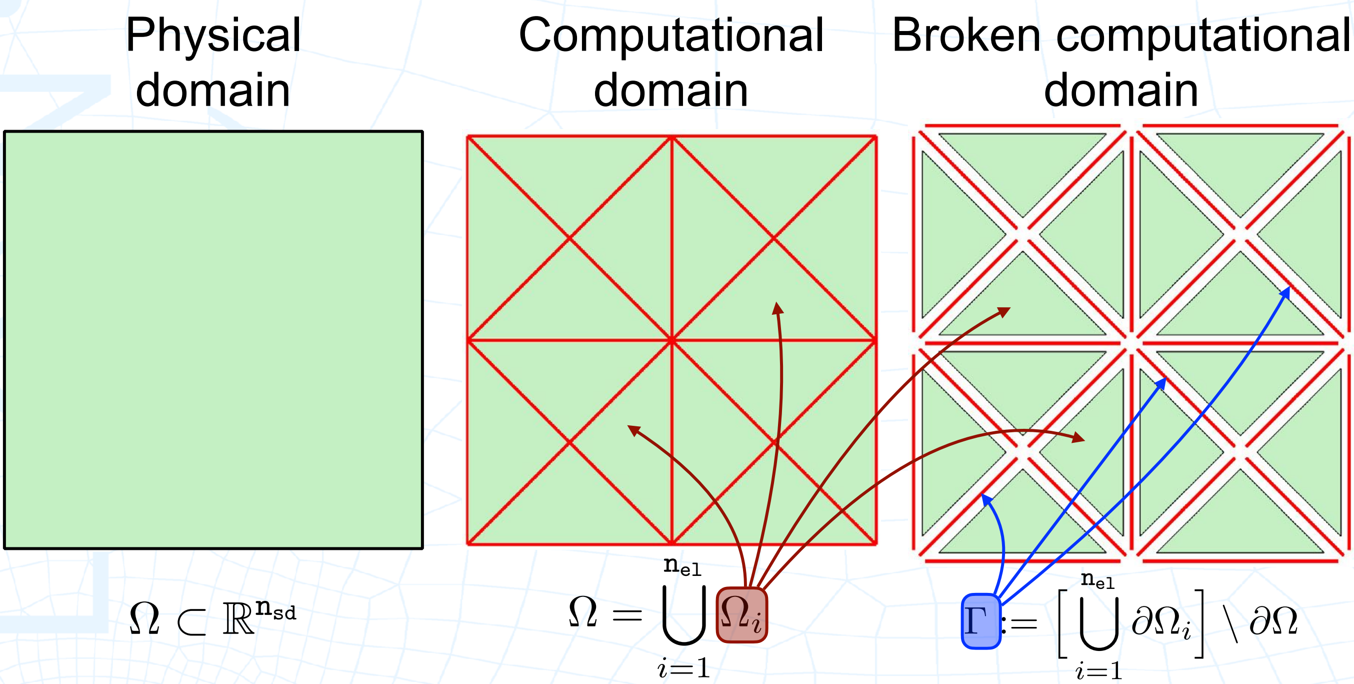
- Open bounded domain: $\Omega \subset \mathbb{R}^{\mathtt{n_{sd}}}$
- External boundary: $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$
- Ω is partitioned in $\mathtt{n_{el}}$ disjoint subdomains Ω_i s.t.

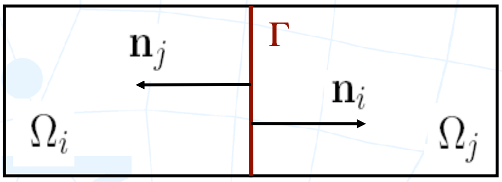
$$\Omega = \bigcup_{i=1}^{\mathtt{n_{el}}} \Omega_i, \qquad \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j$$

with boundaries $\partial\Omega_i$, which define an internal interface Γ

$$\Gamma := \left[\bigcup_{i=1}^{\mathtt{n_{el}}} \partial\Omega_i \right] \setminus \partial\Omega$$

Definitions





Notation

[Montlaur, Fernández-Méndez, Huerta. IJNMF 2008]

- ***Jump*** **operator:**

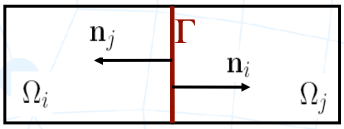
$$[\![u\mathbf{n}]\!] = \begin{cases} u_i\mathbf{n}_i + u_j\mathbf{n}_j & \text{on } \Gamma \\ u\mathbf{n} & \text{on } \partial\Omega \end{cases} \qquad \text{for scalars}$$

$$[\![\boldsymbol{\sigma}\cdot\mathbf{n}]\!] = \begin{cases} \boldsymbol{\sigma}_i\cdot\mathbf{n}_i + \boldsymbol{\sigma}_j\cdot\mathbf{n}_j & \text{on } \Gamma \\ \boldsymbol{\sigma}\cdot\mathbf{n} & \text{on } \partial\Omega \end{cases} \qquad \text{for vectors}$$

- ***Mean*** **operator:**

$$\{u\} = \begin{cases} \frac{1}{2}(u_i + u_j) & \text{on } \Gamma \\ u & \text{on } \partial\Omega \end{cases} \qquad \text{for scalars}$$

$$\{\boldsymbol{\sigma}\} = \begin{cases} \frac{1}{2}(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j) & \text{on } \Gamma \\ \boldsymbol{\sigma} & \text{on } \partial\Omega \end{cases} \qquad \text{for vectors}$$



A useful identity

$$\sum_{e=1}^{\mathtt{n_{el}}} \int_{\partial\Omega_e} \alpha\boldsymbol{w}\cdot\boldsymbol{n}\,d\Gamma = \int_\Gamma \Big([\![\alpha\boldsymbol{n}]\!]\cdot\{\boldsymbol{w}\} + \{\alpha\}[\![\boldsymbol{w}\cdot\boldsymbol{n}]\!] \Big) d\Gamma + \int_{\partial\Omega} \alpha\boldsymbol{w}\cdot\boldsymbol{n}\,d\Gamma$$

- Assume $\mathtt{n_{el}} = 2$ and denote the two elements as Ω_i and Ω_j.
- It follows that $\Omega = \Omega_i \cup \Omega_j$ and $\Gamma = \partial\Omega_i \cap \partial\Omega_j$
  $\partial\Omega = (\partial\Omega_i \cup \partial\Omega_j) \setminus \Gamma$

$$\sum_{e=i,j} \int_{\partial\Omega_e} \alpha\boldsymbol{w}\cdot\boldsymbol{n}\,d\Gamma = \int_{\partial\Omega_i\cap\partial\Omega_j} \Big(\alpha_i\boldsymbol{w}_i\cdot\boldsymbol{n}_i + \alpha_j\boldsymbol{w}_j\cdot\boldsymbol{n}_j \Big) d\Gamma + \int_{(\partial\Omega_i\cup\partial\Omega_j)\setminus\Gamma} \alpha\boldsymbol{w}\cdot\boldsymbol{n}\,d\Gamma$$

A useful identity

- Show that

$$\int_{\partial\Omega_i\cap\partial\Omega_j}\Big(\alpha_i \boldsymbol{w}_i\cdot\boldsymbol{n}_i+\alpha_j \boldsymbol{w}_j\cdot\boldsymbol{n}_j\Big)d\Gamma \overset{?}{=} \int_{\partial\Omega_i\cap\partial\Omega_j}\Big([\![\alpha \boldsymbol{n}]\!]\cdot\{\boldsymbol{w}\}+\{\alpha\}[\![\boldsymbol{w}\cdot\boldsymbol{n}]\!]\Big)d\Gamma$$

$$\int_{\partial\Omega_i\cap\partial\Omega_j}\Big([\![\alpha \boldsymbol{n}]\!]\cdot\{\boldsymbol{w}\}+\{\alpha\}[\![\boldsymbol{w}\cdot\boldsymbol{n}]\!]\Big)d\Gamma$$

$$=\int_{\partial\Omega_i\cap\partial\Omega_j}\Big(\tfrac{1}{2}(\alpha_i\boldsymbol{n}_i+\alpha_j\boldsymbol{n}_j)\cdot(\boldsymbol{w}_i+\boldsymbol{w}_j)+\tfrac{1}{2}(\alpha_i+\alpha_j)(\boldsymbol{w}_i\cdot\boldsymbol{n}_i+\boldsymbol{w}_j\cdot\boldsymbol{n}_j)\Big)d\Gamma$$

$$=\int_{\partial\Omega_i\cap\partial\Omega_j}\Big(\alpha_i\boldsymbol{w}_i\cdot\boldsymbol{n}_i+\alpha_j\boldsymbol{w}_j\cdot\boldsymbol{n}_j\Big)d\Gamma$$

$\boldsymbol{n}_j=-\boldsymbol{n}_i$

$$+\frac{1}{2}\int_{\partial\Omega_i\cap\partial\Omega_j}\Big(\alpha_i\boldsymbol{w}_j\cdot\boldsymbol{n}_i+\alpha_j\boldsymbol{w}_i\cdot\boldsymbol{n}_j+\alpha_i\boldsymbol{w}_j\cdot\boldsymbol{n}_j+\alpha_j\boldsymbol{w}_i\cdot\boldsymbol{n}_i\Big)d\Gamma$$

A review of Discontinuous Galerkin methods

- DG methods originally proposed for pure transport problems. [Reed, Hill. Tech. Report LA-UR-73-479 (1973)]
- In Navier-Stokes equations, convection-diffusion problems, Euler equations with artificial viscosity for shock capturing, it is crucial to properly discretise self-adjoint operators as well.

$$\mathbf{U}_t+\nabla\cdot\mathbf{F}(\mathbf{U})-\nabla\cdot\big(\varepsilon\nabla\mathbf{U}\big)=\mathbf{0}$$

How to treat the self-adjoint operator with a DG formulation?

Interior Penalty Method (IPM)
Local Discontinuous Galerkin (LDG)
...

Interior Penalty Method (IPM)

Douglas N. Arnold (1982)

- Model problem: **Poisson equation**

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \end{cases}$$

Physical domain

-

Interior Penalty Method (IPM)

Douglas N. Arnold (1982)

- Model problem: **Poisson equation**

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \end{cases}$$

- Model problem over the **BROKEN** computational domain

$$\begin{cases} -\Delta u = f & \text{in } \Omega_e, \quad \text{for } e = 1, \ldots, n_{\texttt{el}} \\ u = g & \text{on } \Gamma_D, \\ [\![u\mathbf{n}]\!] = 0 & \text{on } \Gamma, \\ [\![\nabla u \cdot \mathbf{n}]\!] = 0 & \text{on } \Gamma, \end{cases}$$

IMPOSE CONTINUITY OF SOLUTION AND FLUXES

Some comments on IPM

Douglas N. Arnold (1982)

- The solution is sought element-by-element.
- Inter-element continuity of the solution and of the fluxes is relaxed.
- The solution and its gradient can be discontinuous across elements.
- The solution does not belong to $\mathcal{H}^1(\Omega)$.

- **Functional space for test and trial functions:** $\mathcal{L}_2(\Omega)$
- The solution is approximated using a polynomial function within each element Ω_e.
- Dirichlet boundary conditions are imposed in a weak sense.

Interior Penalty Method (IPM)

Douglas N. Arnold (1982)

$$\begin{cases} -\Delta u = f & \text{in } \Omega_e, \text{ for } e = 1, \dots, n_{\texttt{el}} \\ u = g & \text{on } \Gamma_D, \\ [\![u\mathbf{n}]\!] = 0 & \text{on } \Gamma, \\ [\![\nabla u \cdot \mathbf{n}]\!] = 0 & \text{on } \Gamma, \end{cases}$$

for simplicity: Dirichlet BC on all external boundary

- Write the weak form in a generic element Ω_e: find $u \in \mathcal{H}^1(\Omega_e)$ such that $u = g$ on $\partial\Omega_e \cap \partial\Omega$ and $\forall v \in \mathcal{H}^1(\Omega_e)$ it holds

$$\int_{\Omega_e} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega - \int_{\partial\Omega_e} (\boldsymbol{\nabla} u \cdot \mathbf{n}) \, v \, d\Gamma = \int_{\Omega_e} f \, v \, d\Omega$$

where **n** is the unit outward normal vector to $\partial\Omega_e$.

IPM weak form

- Sum the element-by-element contributions:

$$\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega - \sum_e \int_{\partial\Omega_e} (\boldsymbol{\nabla} u \cdot \mathbf{n}) \, v \, d\Gamma = \int_\Omega f \, v \, d\Omega$$

Useful identity

$$\sum_{e=1}^{\mathtt{n_{el}}} \int_{\partial\Omega_e} \alpha \boldsymbol{w} \cdot \boldsymbol{n} d\Gamma = \int_\Gamma \Big([\![\alpha \boldsymbol{n}]\!] \cdot \{ \boldsymbol{w} \} + \{ \alpha \} [\![\boldsymbol{w} \cdot \boldsymbol{n}]\!] \Big) d\Gamma + \int_{\partial\Omega} \alpha \boldsymbol{w} \cdot \boldsymbol{n} d\Gamma$$

Γ or Γ_{int} is the union of all interior edges/faces

$\partial\Omega$ is the union of all exteriors edges/faces, which can be split in Dirichlet, Γ_D, and Neumann, Γ_N, boundaries

IPM weak form

- Summing over all elements

$$\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega - \sum_e \int_{\partial\Omega_e} (\boldsymbol{\nabla} u \cdot \mathbf{n}) \, v \, d\Gamma = \int_\Omega f \, v \, d\Omega$$

Useful identity

$$\sum_{e=1}^{\mathtt{n_{el}}} \int_{\partial\Omega_e} \alpha \boldsymbol{w} \cdot \boldsymbol{n} d\Gamma = \int_\Gamma \Big([\![\alpha \boldsymbol{n}]\!] \cdot \{ \boldsymbol{w} \} + \{ \alpha \} [\![\boldsymbol{w} \cdot \boldsymbol{n}]\!] \Big) d\Gamma + \int_{\partial\Omega} \alpha \boldsymbol{w} \cdot \boldsymbol{n} d\Gamma$$

$\partial\Omega = \Gamma_D$ (Only Dir. BC)

$$\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega - \int_\Gamma \left([\![v \mathbf{n}]\!] \cdot \{ \boldsymbol{\nabla} u \} + \{ v \} [\![\boldsymbol{\nabla} u \cdot \mathbf{n}]\!] \right) d\Gamma - \int_{\Gamma_D} v \boldsymbol{n} \cdot \boldsymbol{\nabla} u \, d\Gamma = \int_\Omega f v \, d\Omega$$

$[\![\boldsymbol{\nabla} u \cdot \mathbf{n}]\!] = 0 \text{ on } \Gamma$ (Flux continuity)

IPM symmetric weak form

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_\Gamma \left([\![v\mathbf{n}]\!] \cdot \{\nabla u\} + \{v\}[\![\nabla u \cdot \mathbf{n}]\!] \right) d\Gamma - \int_{\Gamma_D} v\boldsymbol{n} \cdot \nabla u \, d\Gamma = \int_\Omega f v \, d\Omega$$

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma \cup \Gamma_D} [\![v\mathbf{n}]\!] \cdot \{\nabla u\} \, d\Gamma = \int_\Omega f v \, d\Omega$$

non-symmetric

- Adding terms to obtain a **consistent** symmetric bilinear form:

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma \cup \Gamma_D} \left([\![v\mathbf{n}]\!] \cdot \{\nabla u\} + [\![u\mathbf{n}]\!] \cdot \{\nabla v\} \right) d\Gamma$$

$$= \int_\Omega f v \, d\Omega - 0 - \int_{\Gamma_D} g \nabla v \cdot \mathbf{n} \, d\Gamma$$

$[\![u\boldsymbol{n}]\!] = \mathbf{0}$ on Γ (Solution continuity)

IPM symmetric weak form

- The bilinear form is symmetric but may be not coercive.
- Add the **consistent** penalty term:

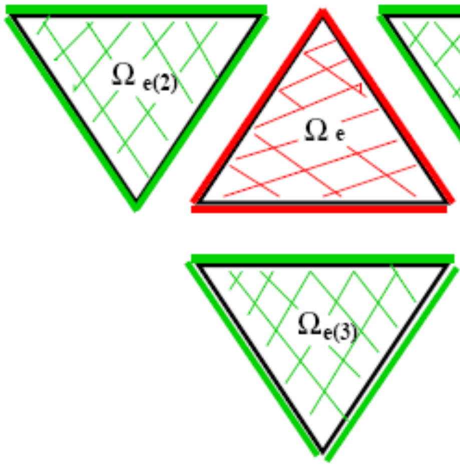
$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_{\Gamma \cup \Gamma_D} \left([\![v\mathbf{n}]\!] \cdot \{\nabla u\} + [\![u\mathbf{n}]\!] \cdot \{\nabla v\} \right) d\Gamma + \int_{\Gamma \cup \Gamma_D} \beta [\![u\mathbf{n}]\!] \cdot [\![v\mathbf{n}]\!] \, d\Gamma$$

$$= \int_\Omega f v \, d\Omega - 0 - \int_{\Gamma_D} g \nabla v \cdot \mathbf{n} \, d\Gamma + 0 + \int_{\Gamma_D} \beta \, g v \, d\Gamma$$

$[\![u\boldsymbol{n}]\!] = \mathbf{0}$ on Γ (Solution continuity)

- The bilinear form is coercive for β large enough. $\beta = \alpha h^{-1}$
- This constant is typical from the weak imposition of essential boundary conditions using Nitsche's method.

IPM stencil in 2D



Only neighboring elements adjacent to Ω_e are used

But ∇u must be calculated on $\partial\Omega_e$

(thus, dependence on all neighboring nodes)

IPM convergence

- Using polynomials of degree p the following optimal rates of convergence are demonstrated for the $\mathcal{L}_2$ norm of

Variable	Order of convergence
u	p+1
∇u	p

- If the penalty parameter is not defined as $\beta = \alpha h^{-1}$ the optimal rate of convergence can be degraded.

Local Discontinuous Galerkin (LDG)

Cockburn and Shu (1998)

- Model problem
$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \end{cases}$$
- DG method based on a mixed formulation (system of first-order PDEs):
$$\begin{cases} \boldsymbol{\sigma} - \boldsymbol{\nabla} u = \mathbf{0} & \text{in } \Omega, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \end{cases}$$

Mixed formulation in broken computational domain

$$\begin{cases} \boldsymbol{\sigma} - \boldsymbol{\nabla} u = \mathbf{0} & \text{in } \Omega, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \end{cases}$$

$$\begin{cases} \boldsymbol{\sigma} - \boldsymbol{\nabla} u = \mathbf{0} & \text{in } \Omega_e, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = f & \text{in } \Omega_e, \\ [\![u \boldsymbol{n}]\!] = \mathbf{0} & \text{on } \Gamma, \\ [\![\boldsymbol{n} \cdot \boldsymbol{\sigma}]\!] = 0 & \text{on } \Gamma, \\ u = g & \text{on } \partial\Omega := \Gamma_D. \end{cases}$$

for $e = 1, \dots, n_{\texttt{el}}$

IMPOSE CONTINUITY OF SOLUTION AND FLUXES

$$\begin{cases} \boldsymbol{\sigma} - \boldsymbol{\nabla} u = \mathbf{0} & \text{in } \Omega_e, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = f & \text{in } \Omega_e, \\ u = g & \text{on } \partial\Omega. \end{cases} \qquad \begin{cases} [\![u\, \boldsymbol{n}]\!] = \mathbf{0} & \text{on } \Gamma, \\ [\![\boldsymbol{n} \cdot \boldsymbol{\sigma}]\!] = 0 & \text{on } \Gamma, \end{cases}$$

- Write the weak form in an element Ω_e: find $u \in \mathcal{L}_2(\Omega_e)$ and $\boldsymbol{\sigma} \in \mathcal{H}(\mathrm{div}, \Omega_e; \mathbb{R}^{\mathrm{n_{sd}}})$ such that $\forall (v, \boldsymbol{\tau}) \in (\mathcal{H}^1(\Omega_e), \mathcal{H}(\mathrm{div}, \Omega_e; \mathbb{R}^{\mathrm{n_{sd}}}))$ it holds

$$\int_{\Omega_e} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, d\Omega + \int_{\Omega_e} u \, \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \, d\Omega - \int_{\partial\Omega_e} \hat{u} \, \mathbf{n} \cdot \boldsymbol{\tau} \, d\Gamma = 0$$

$$\int_{\Omega_e} \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v \, d\Omega - \int_{\partial\Omega_e} \hat{\boldsymbol{\sigma}} \cdot \mathbf{n} \, v \, d\Gamma = \int_{\Omega_e} f \, v \, d\Omega$$

$$\hat{u} := \{u\} + \mathbf{C_{12}} \cdot [\![u\mathbf{n}]\!]$$
$$\hat{\boldsymbol{\sigma}} := \{\boldsymbol{\sigma}\} - \mathbf{C_{12}} [\![\boldsymbol{\sigma} \cdot \mathbf{n}]\!] - C_{11} [\![u\mathbf{n}]\!]$$

(Numerical fluxes)

with $C_{12} = \frac{1}{2}(S_{ij}\mathbf{n}_i + S_{ji}\mathbf{n}_j)$ and a switch such that $S_{ij} + S_{ji} = 1$

Some comments on LDG

- The flux of the solution u does not depend on $\boldsymbol{\sigma}$ **LOCAL DG**
- To guarantee consistency, the fluxes are such that

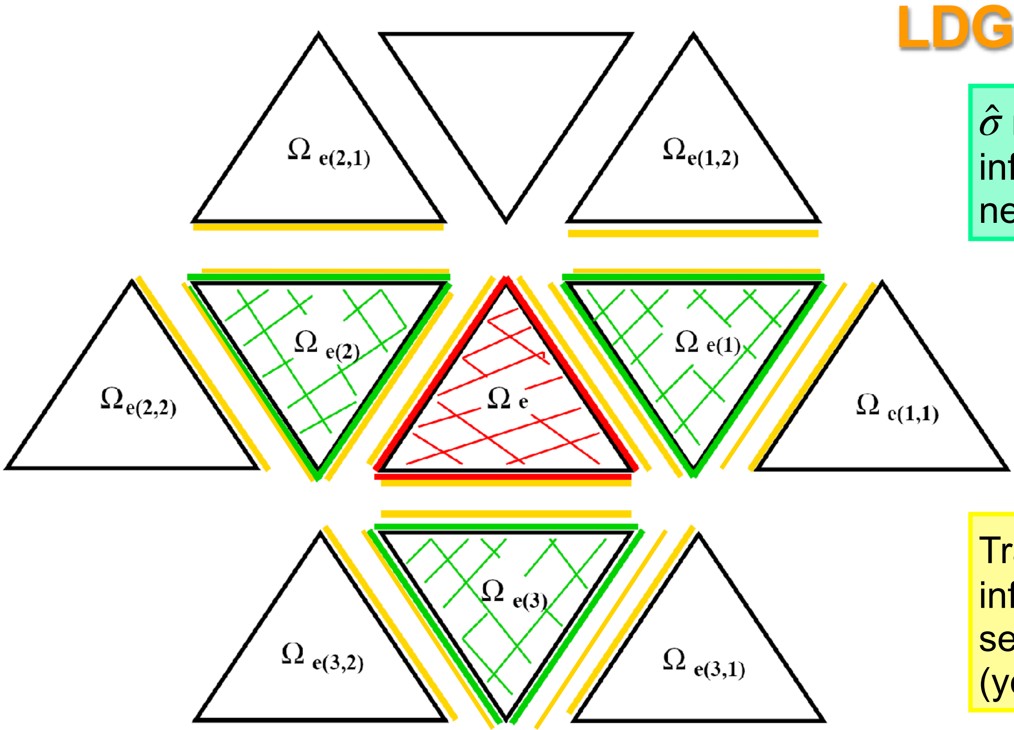
$$\hat{u}(u) = u_{|\Gamma} \quad \text{(trace of the solution)}$$
$$\hat{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, u) = \boldsymbol{\sigma}_{|\Gamma}$$

- Conservation properties are fulfilled noticing that

$$\begin{cases} [\![u\, \boldsymbol{n}]\!] = \mathbf{0} \\ [\![\boldsymbol{n} \cdot \boldsymbol{\sigma}]\!] = 0 \end{cases} \longleftrightarrow \begin{array}{l} [\![\hat{u}\mathbf{n}]\!] = \hat{u}_i \mathbf{n}_i + \hat{u}_j \mathbf{n}_j = \mathbf{0} \\ [\![\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}]\!] = \hat{\boldsymbol{\sigma}}_i \cdot \mathbf{n}_i + \hat{\boldsymbol{\sigma}}_j \cdot \mathbf{n}_j = 0 \end{array}$$

- Dirichlet conditions are imposed in a weak sense through numerical fluxes.

LDG stencil in 2D



$\hat{\sigma}$ requires the information of the first neighbors (green)

Trace $\hat{u}$ involves the information of the second neighbors (yellow)

LDG convergence

- Using polynomials of degree p the following optimal rates of convergence are demonstrated for the $\mathcal{L}_2$ norm of

Variable	Order of convergence
u	p+1
$\boldsymbol{\sigma}$	p

- The optimal order of convergence in the $\mathcal{L}_2$ norm is obtained when the parameter C_{11} is mesh-dependent ($C_{11} = \alpha h^{-1}$ like the penalty parameter of the IPM).
- If C_{11} is constant the order is not optimal: $\mathrm{p} + 1/2$.

DG unified analysis for self-adjoint operators

[Arnold, Brezzi, Cockburn, Marini. SINUM 2002]

Method	$\widehat{u}_K$	$\widehat{\sigma}_K$
Bassi–Rebay [9]	$\{u_h\}$	$\{\sigma_h\}$
Brezzi et al. [18]	$\{u_h\}$	$\{\sigma_h\} - \alpha_r([\![u_h]\!])$
LDG [35]	$\{u_h\} - \beta \cdot [\![u_h]\!]$	$\{\sigma_h\} + \beta[\![\sigma_h]\!] - \alpha_j([\![u_h]\!])$
IP [43]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_j([\![u_h]\!])$
Bassi et al. [10]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_r([\![u_h]\!])$
Baumann–Oden [12]	$\{u_h\} + n_K \cdot [\![u_h]\!]$	$\{\nabla_h u_h\}$
NIPG [53]	$\{u_h\} + n_K \cdot [\![u_h]\!]$	$\{\nabla_h u_h\} - \alpha_j([\![u_h]\!])$
Babuška–Zlámal [6]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_j([\![u_h]\!])$
Brezzi et al. [19]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_r([\![u_h]\!])$

Some remarks

- There is a large family of methods following the DG rationale for elliptic operators.
- Transmission conditions are used to weakly impose inter-element continuity of the solution and of the fluxes.
- Mixed formulations can also be employed to improve robustness.
- Freedom to choose interpolation functions element-by-element, numerical flux stabilisation, local conservation, …

Is the overhead cost of node duplication at edges/faces justified?

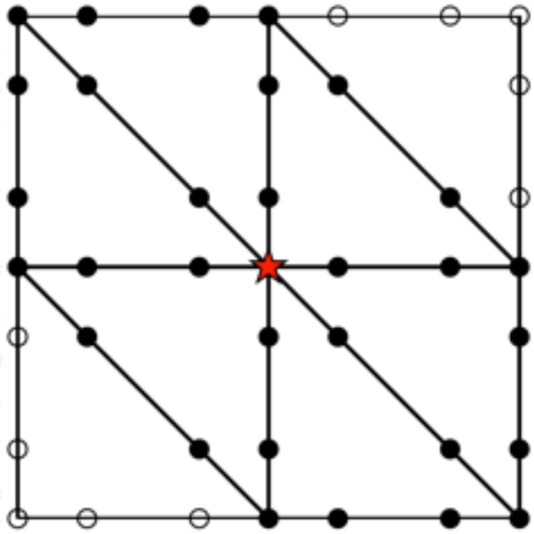
Hybridisable Discontinuous Galerkin formulation

Useful tutorials

- R. Sevilla and A. Huerta, "Tutorial on Hybridizable Discontinuous Galerkin (HDG) for second-order elliptic problems", in *Advanced Finite Element Technologies*, CISM International Centre for Mechanical Sciences, Vol 566, pp. 105-129, Springer, Eds. J. Schröder and P. Wriggers (2016). DOI: 10.1007/978-3-319-31925-4_5
- M. Giacomini, R. Sevilla and A. Huerta, "Tutorial on Hybridizable Discontinuous Galerkin (HDG) Formulation for Incompressible Flow Problems", in *Modeling in Engineering Using Innovative Numerical Methods for Solids and Fluids*, CISM International Centre for Mechanical Sciences, Vol. 599, pp.163-201, Springer, Eds. L. De Lorenzis and A. Düster (2020). DOI: 10.1007/978-3-030-37518-8_5
- M. Giacomini, R. Sevilla and A. Huerta, "HDGlab: An Open-Source Implementation of the Hybridisable Discontinuous Galerkin Method in MATLAB", Archives of Computational Methods in Engineering, Vol. 28, Issue 3, pp. 1941-1986 (2021). DOI: 10.1007/s11831-020-09502-5

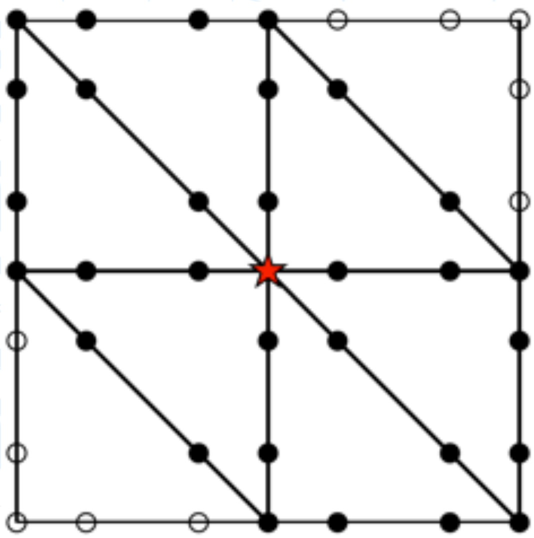
Degrees of freedom

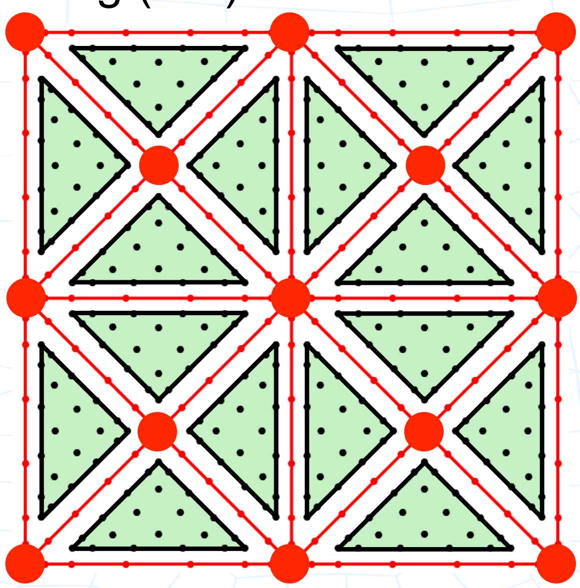
Continuous Galerking (**CG**)



Degrees of freedom

Continuous Galerking (**CG**)

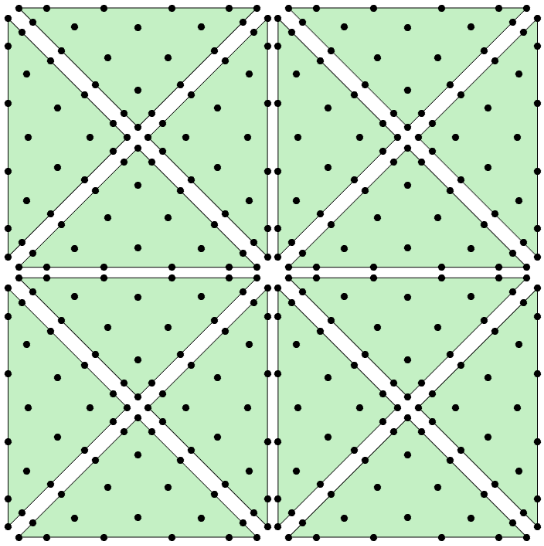




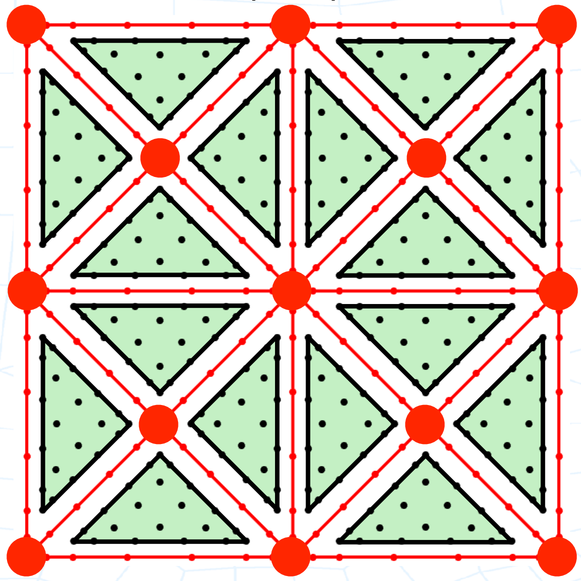
Interior nodes are decoupled by static condensation

Degrees of freedom

Discontinuous Galerkin (**DG**)



(**CG**)



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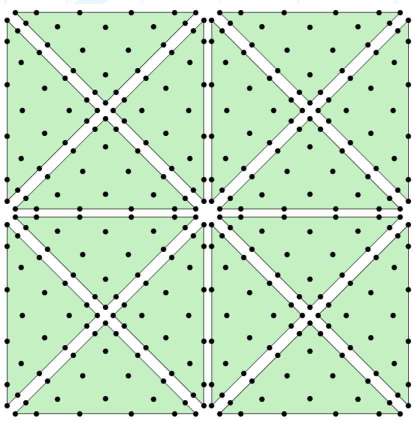
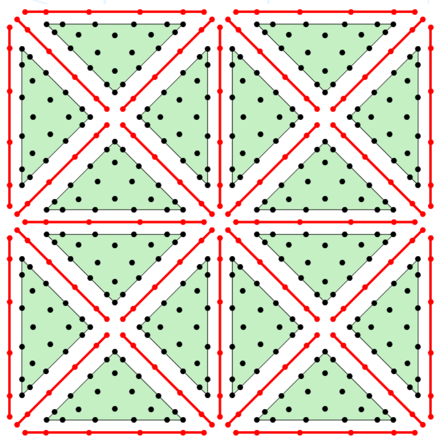
Degrees of freedom

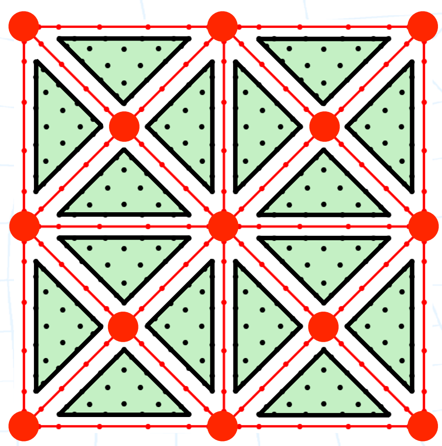
(**DG**)



HDG



(**CG**)



Hybridisable Discontinuous Gakerkin (**HDG**) method proposed by Bernardo Cockburn and Jay Gopalakrishnan in early 2000's

What is a hybrid method?

"we may define more generally as a ***hybrid method*** any finite element method based on a formulation where **one unknown is a function**, or some of its derivatives, **on a set Ω**, and **the other unknown is the trace of** some of its derivatives of the same function, or **the function itself, along the boundaries of the set K.**"

discretisation

P. Ciarlet, *The finite element method for elliptic problems*, SIAM, 2002.

The origins of HDG

- Seminal works: [Guyan AIAA 1965] [Fraeijs de Veubeke 1965]
- Element-by-element discontinuous approximations:
  - ✓ hybrid/hybridised DG: [Egger, Schöberl 2009] [Egger, Waluga 2012] [Oikawa 2015/16]
  - ✓ hybridisable DG (HDG) [Cockburn…] [BC, Gopalakrishnan 2009] [Nguyen Peraire BC 2009]
  - ✓ hybrid high order (HHO) [Di Pietro, Ern … 2014/2015]
- See a recent review in

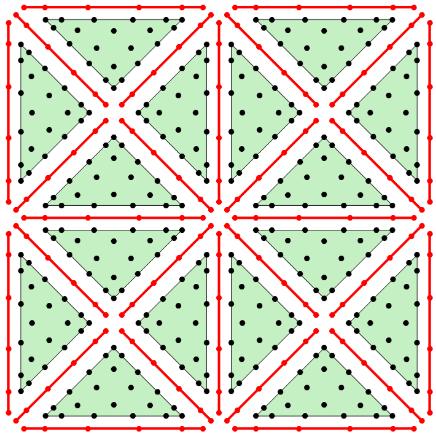
[Giacomini, Sevilla, AH **HDGlab** Arch. Comp. Meth. Eng. 2021]

HDG rationale

Cockburn and Gopalakrishnan (2005)

1) The solution is **discontinuous element-by-element**.
2) The **solution** and its **gradient** are approximated in **each element** using piecewise polynomial functions.
3) The **solution** is approximated on **each face** using polynomial functions.
4) Equal-order approximation is applied to **all the variables**.
5) Elemental unknowns are eliminated via **static condensation**.

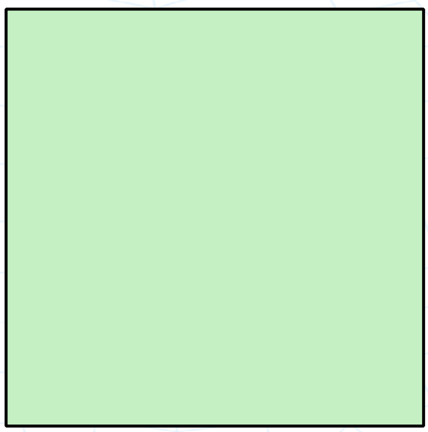
HDG



Hybridisable Discontinuous Galerkin

- Model problem: solve BVP in Ω

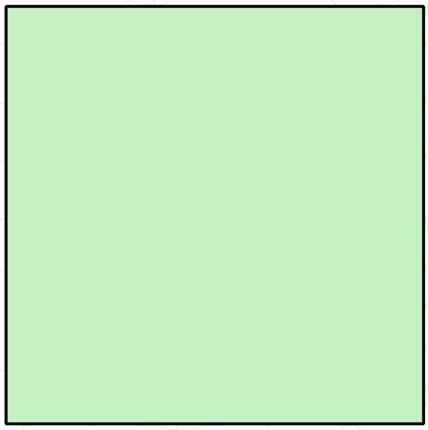
$$\begin{cases} -\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} u = s & \text{in } \Omega, \\ u = u_D & \text{on } \Gamma_D, \\ \boldsymbol{n}\cdot\boldsymbol{\nabla} u = t & \text{on } \Gamma_N, \end{cases}$$



Mixed formulation

- Introduce a new unknown representing the gradient of u:

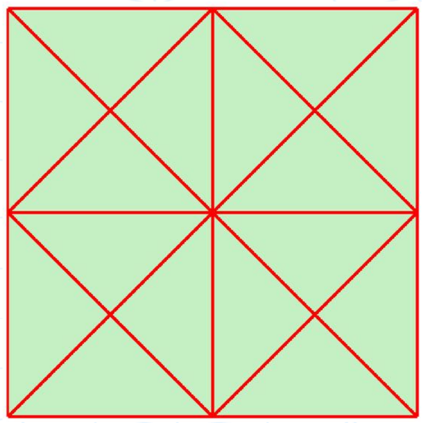
$$\begin{cases} \boldsymbol{\sigma} - \boldsymbol{\nabla} u = \mathbf{0} & \text{in } \Omega, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = s & \text{in } \Omega, \\ u = u_D & \text{on } \Gamma_D, \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} = -t & \text{on } \Gamma_N. \end{cases}$$



This improves stability of the discontinuous method

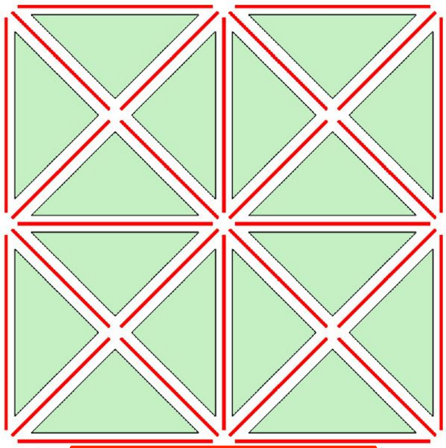
Mixed formulation in the broken domain

- Define the *broken computational domain* where Ω is partitioned in disjoint subdomains $\Omega_e, \quad e = 1, \ldots, \texttt{n}_{\texttt{el}}$



Mixed formulation in the broken domain

- Define the *broken computational domain* where Ω is partitioned in disjoint subdomains $\Omega_e\,,\quad e = 1,\ldots,\mathtt{n_{el}}$ with interior skeleton Γ.



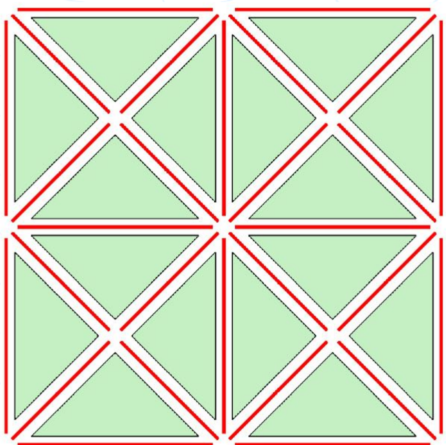
$$\Gamma := \Big[\bigcup_{i=1}^{\mathtt{n_{el}}} \partial\Omega_i\Big] \setminus \partial\Omega$$

Mixed formulation in the broken domain

- Define the *broken computational domain* where Ω is partitioned in disjoint subdomains $\Omega_e\,,\quad e = 1,\ldots,\mathtt{n_{el}}$ with interior skeleton Γ.
- Rewrite the equivalent problem:

$$\begin{cases} \boldsymbol{\sigma}_e - \boldsymbol{\nabla} u_e = \mathbf{0} & \text{in } \Omega_e, \text{ for } e = 1,\ldots,\mathtt{n_{el}}, \\ -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_e = s & \text{in } \Omega_e, \text{ for } e = 1,\ldots,\mathtt{n_{el}}, \\ u = u_D & \text{on } \Gamma_D, \\ \boldsymbol{n}\cdot\boldsymbol{\sigma} = -t & \text{on } \Gamma_N, \\ [\![u\,\boldsymbol{n}]\!] = \mathbf{0} & \text{on } \Gamma, \\ [\![\boldsymbol{n}\cdot\boldsymbol{\sigma}]\!] = 0 & \text{on } \Gamma. \end{cases}$$

IMPOSE CONTINUITY OF SOLUTION AND FLUXES

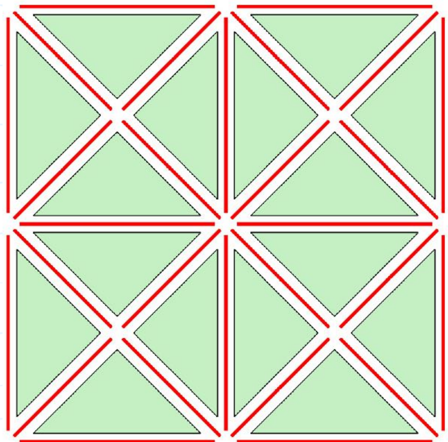


$$\Gamma := \Big[\bigcup_{i=1}^{\mathtt{n_{el}}} \partial\Omega_i\Big] \setminus \partial\Omega$$

Elemental and face-based problems

- On the broken computational domain, split the BVP into:
  1) element-by-element problems
  2) face problems

$$\begin{cases} \boldsymbol{\sigma}_e - \boldsymbol{\nabla} u_e = \mathbf{0} & \text{in } \Omega_e, \text{ for } e = 1, \dots, \mathtt{n_{el}}, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_e = s & \text{in } \Omega_e, \text{ for } e = 1, \dots, \mathtt{n_{el}}, \\ u = u_D & \text{on } \Gamma_D, \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} = -t & \text{on } \Gamma_N, \\ [\![u \, \boldsymbol{n}]\!] = \mathbf{0} & \text{on } \Gamma, \\ [\![\boldsymbol{n} \cdot \boldsymbol{\sigma}]\!] = 0 & \text{on } \Gamma. \end{cases}$$



Elemental and face-based problems

- On the broken computational domain, split the BVP into:
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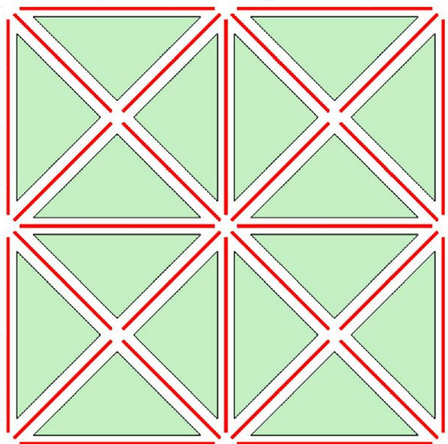
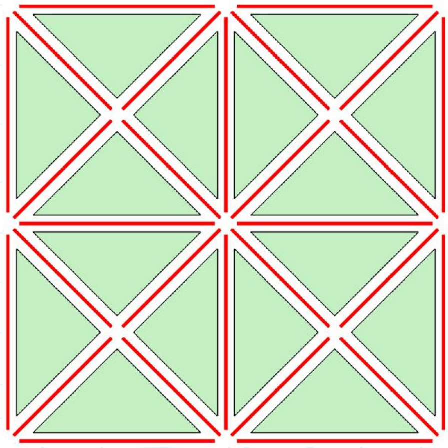
$$\begin{cases} \boldsymbol{\sigma}_e - \boldsymbol{\nabla} u_e = \mathbf{0} & \text{in } \Omega_e, \text{ for } e = 1, \dots, \mathtt{n_{el}}, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_e = s & \text{in } \Omega_e, \text{ for } e = 1, \dots, \mathtt{n_{el}}, \\ u = u_D & \text{on } \Gamma_D, \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} = -t & \text{on } \Gamma_N, \\ [\![u \, \boldsymbol{n}]\!] = \mathbf{0} & \text{on } \Gamma, \\ [\![\boldsymbol{n} \cdot \boldsymbol{\sigma}]\!] = 0 & \text{on } \Gamma. \end{cases}$$



Elemental problems

1) Solve element-by-element for $(u_e, \boldsymbol{\sigma}_e), \; e = 1, \dots, \texttt{n}_{\texttt{el}}$

$$
\begin{cases}
\boldsymbol{\sigma}_e - \boldsymbol{\nabla} u_e = \mathbf{0} & \text{in } \Omega_e, \\
-\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_e = s & \text{in } \Omega_e, \\
u_e = u_D & \text{on } \partial\Omega_e \cap \Gamma_D, \\
? & \text{on } \partial\Omega_e \setminus \Gamma_D.
\end{cases}
$$

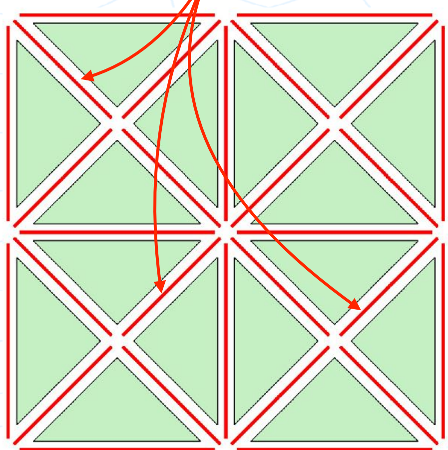


Elemental problems

1) Assume $\hat{u} \in \mathcal{L}_2(\Gamma \cup \Gamma_N)$ given and solve element-by-element for $(u_e, \boldsymbol{\sigma}_e), \; e = 1, \dots, \texttt{n}_{\texttt{el}}$

$$
\begin{cases}
\boldsymbol{\sigma}_e - \boldsymbol{\nabla} u_e = \mathbf{0} & \text{in } \Omega_e, \\
-\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_e = s & \text{in } \Omega_e, \\
u_e = u_D & \text{on } \partial\Omega_e \cap \Gamma_D, \\
u_e = \hat{u} & \text{on } \partial\Omega_e \setminus \Gamma_D.
\end{cases}
$$

The hybrid variable!



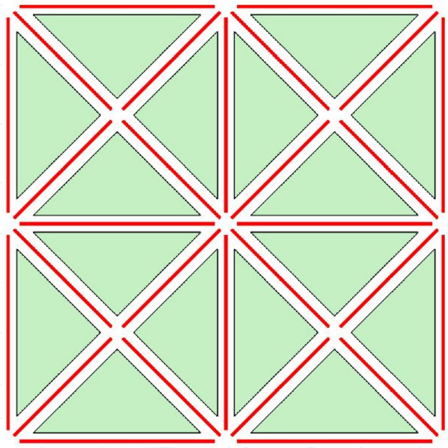
Solve for $(u_e, \boldsymbol{\sigma}_e)$ as a function of $\hat{u}$

(Static condensation)

Face-based problem

1) Assume $\hat{u} \in \mathcal{L}_2(\Gamma \cup \Gamma_N)$ given and solve element-by-element for $(u_e, \boldsymbol{\sigma}_e),\ e = 1, \dots, \mathtt{n_{el}}$

$$\begin{cases} \boldsymbol{\sigma}_e - \boldsymbol{\nabla} u_e = \mathbf{0} & \text{in } \Omega_e, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_e = s & \text{in } \Omega_e, \\ u_e = u_D & \text{on } \partial\Omega_e \cap \Gamma_D, \\ u_e = \hat{u} & \text{on } \partial\Omega_e \setminus \Gamma_D. \end{cases}$$



2) On the interior skeleton (+*Neumann*), solve for the hybrid variable $\hat{u}$ imposing:

$$\begin{cases} \boldsymbol{n} \cdot \boldsymbol{\sigma} = -t & \text{on } \Gamma_N, \\ [\![u\, \boldsymbol{n}]\!] = \mathbf{0} & \text{on } \Gamma, \\ [\![\boldsymbol{n} \cdot \boldsymbol{\sigma}]\!] = 0 & \text{on } \Gamma. \end{cases}$$

HDG weak form

- To write the weak forms:
  Define the $\mathcal{L}_2$ scalar products for scalars $(u, v)_D = \int_D u\, v\, d\Omega$
  and vectors $(\boldsymbol{u}, \boldsymbol{v})_D = \int_D \boldsymbol{u} \cdot \boldsymbol{v}\, d\Omega$ in any subdomain *D*
  or any boundary $S \subset \Gamma \cup \partial\Omega$
  $< u, v >_S = \int_S u\, v\, d\Gamma$ and $< \boldsymbol{u}, \boldsymbol{v} >_S = \int_S \boldsymbol{u} \cdot \boldsymbol{v}\, d\Gamma$
  And the weak local and global problem on the spaces:

$$\mathcal{V}^h(\Omega) := \{ v \in \mathcal{L}_2(\Omega) : v|_{\Omega_e} \in \mathcal{P}^{\mathrm{p}}(\Omega_e)\ \forall \Omega_e,\ e = 1, \dots, \mathtt{n_{el}} \}$$

$$\hat{\mathcal{V}}^h(S) := \{ \hat{v} \in \mathcal{L}_2(S) : \hat{v}|_{\Gamma_i} \in \mathcal{P}^{\mathrm{p}}(\Gamma_i)\ \forall \Gamma_i \subset S \subseteq \Gamma \cup \partial\Omega \}$$

HDG weak form: local problem

- **Local problem:** find $(u_e, \boldsymbol{\sigma}_e),\ e = 1, \dots, \mathtt{n_{el}}$ by solving

$$\begin{cases} \boldsymbol{\sigma}_e - \boldsymbol{\nabla} u_e = \mathbf{0} & \text{in } \Omega_e, \\ -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_e = s & \text{in } \Omega_e, \\ u_e = u_D & \text{on } \partial\Omega_e \cap \Gamma_D, \\ u_e = \hat{u} & \text{on } \partial\Omega_e \setminus \Gamma_D. \end{cases}$$

- Given u_D on Γ_D and $\hat{u}$ on $\Gamma \cup \Gamma_N$, find $(u_e^h, \boldsymbol{\sigma}_e^h) \in \mathcal{V}^h(\Omega_e) \times [\mathcal{V}^h(\Omega_e)]^{\mathtt{n_{sd}}}$

$$(-\boldsymbol{w}, \boldsymbol{\sigma}_e^h)_{\Omega_e} + (\boldsymbol{\nabla} \cdot \boldsymbol{w}, u_e^h)_{\Omega_e} = \langle \boldsymbol{n}_e \cdot \boldsymbol{w}, u_D \rangle_{\partial\Omega_e \cap \Gamma_D} + \langle \boldsymbol{n}_e \cdot \boldsymbol{w}, \hat{u}^h \rangle_{\partial\Omega_e \setminus \Gamma_D}$$

$$-(\boldsymbol{\nabla} v, \boldsymbol{\sigma}_e^h)_{\Omega_e} + \langle v, \boldsymbol{n}_e \cdot \widehat{\boldsymbol{\sigma}}_e^h \rangle_{\partial\Omega_e} = (v, s)_{\Omega_e}$$

for all $(v, \boldsymbol{w}) \in \mathcal{V}^h(\Omega_e) \times [\mathcal{V}^h(\Omega_e)]^{\mathtt{n_{sd}}}$

Numerical fluxes

Stabilisation coefficient

$$\boldsymbol{n}_e \cdot \widehat{\boldsymbol{\sigma}}_e^h := \begin{cases} \boldsymbol{n}_e \cdot \boldsymbol{\sigma}_e^h + \tau (u_e - u_D) & \text{on } \partial\Omega \cap \Gamma_D, \\ \boldsymbol{n}_e \cdot \boldsymbol{\sigma}_e^h + \tau (u_e - \hat{u}) & \text{elsewhere,} \end{cases}$$

HDG weak form: local problem

- Exploit the definition of the numerical fluxes and integrate by parts again the left-hand side of the 2nd equation to retrieve a symmetric formulation, the discrete weak problem becomes: find $(u_e^h, \boldsymbol{\sigma}_e^h) \in \mathcal{V}^h(\Omega_e) \times [\mathcal{V}^h(\Omega_e)]^{\mathtt{n_{sd}}}$ s.t. for all $(v, \boldsymbol{w}) \in \mathcal{V}^h(\Omega_e) \times [\mathcal{V}^h(\Omega_e)]^{\mathtt{n_{sd}}}$

$$(-\boldsymbol{w}, \boldsymbol{\sigma}_e^h)_{\Omega_e} + (\boldsymbol{\nabla} \cdot \boldsymbol{w}, u_e^h)_{\Omega_e} = \langle \boldsymbol{n}_e \cdot \boldsymbol{w}, u_D \rangle_{\partial\Omega_e \cap \Gamma_D} + \langle \boldsymbol{n}_e \cdot \boldsymbol{w}, \hat{u}^h \rangle_{\partial\Omega_e \setminus \Gamma_D}$$

$$(v, \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_e^h)_{\Omega_e} + \langle v, \tau u_e^h \rangle_{\partial\Omega_e} = (v, s)_{\Omega_e} + \langle v, \tau u_D \rangle_{\partial\Omega_e \cap \Gamma_D} + \langle v, \tau \hat{u}^h \rangle_{\partial\Omega_e \setminus \Gamma_D}$$

- **Theorem (Well posedness)** [Cockburn et al., SINUM (2009)]
The local solver on Ω_i for each element $i = 1, \dots \mathtt{n_{el}}$ is well defined if $\tau_i > 0$ on $\partial\Omega_i$ and $\boldsymbol{\nabla}\mathcal{V}^h(\Omega_i) \subset [\mathcal{V}^h(\Omega_i)]^{\mathtt{n_{sd}}}$.

We still need to determine $\hat{u}^h$, which is an unknown!!!

HDG weak form: global problem

- **Global problem:** solve for the hybrid variable $\hat{u}$ imposing

$$\begin{cases} \boldsymbol{n}\cdot\boldsymbol{\sigma} = -t & \text{on } \Gamma_N, \\ [\![\boldsymbol{n}\cdot\boldsymbol{\sigma}]\!] = 0 & \text{on } \Gamma. \end{cases}$$

seek $\hat{u}^h \in \hat{\mathcal{V}}^h(\Gamma\cup\Gamma_N)$ such that for all $\hat{v} \in \hat{\mathcal{V}}^h(\Gamma\cup\Gamma_N)$

$$\sum_{e=1}^{\mathtt{n_{el}}} \langle \hat{v}, \boldsymbol{n}_e\cdot\widehat{\boldsymbol{\sigma}}_e^h\rangle_{\partial\Omega_e\setminus\partial\Omega} + \sum_{e=1}^{\mathtt{n_{el}}} \langle \hat{v}, \boldsymbol{n}_e\cdot\widehat{\boldsymbol{\sigma}}_e^h + t\rangle_{\partial\Omega_e\cap\Gamma_N} = 0$$

Numerical fluxes

$(u_e^h, \boldsymbol{\sigma}_e^h)$ are written in terms of $\hat{u}^h$ solving the local problems

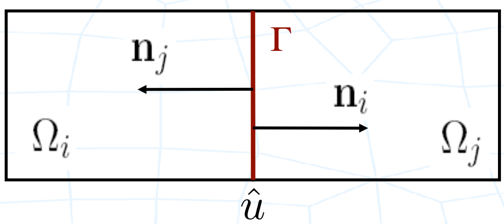
$$\sum_{e=1}^{\mathtt{n_{el}}} \Big\{ \langle \hat{v}, \boldsymbol{n}_e\cdot\boldsymbol{\sigma}_e^h\rangle_{\partial\Omega_e\setminus\Gamma_D} + \sum_{e=1}^{\mathtt{n_{el}}} \langle \hat{v}, \tau u_e^h\rangle_{\partial\Omega_e\setminus\Gamma_D} - \sum_{e=1}^{\mathtt{n_{el}}} \langle \hat{v}, \tau \hat{u}^h\rangle_{\partial\Omega_e\setminus\Gamma_D} \Big\} = -\sum_{e=1}^{\mathtt{n_{el}}} \langle \hat{v}, t\rangle_{\partial\Omega_e\cap\Gamma_N}$$

Stabilisation coefficient

Some comments on HDG

- The solution of the element-by-element problems is equivalent to a static condensation in CG.
- Inter-element continuity of the solution is automatically fulfilled owing to the uniqueness of the hybrid variable on the element boundary:

$$[\![u\boldsymbol{n}]\!] = \hat{u}\boldsymbol{n}_i + \hat{u}\boldsymbol{n}_j = \boldsymbol{0} \text{ on } \Gamma$$



- Dirichlet conditions are imposed in a weak sense through numerical fluxes.

Superconvergence

- Optimal convergence rates for **both** approximations u^h and $\boldsymbol{\sigma}^h$ in $\mathcal{L}_2$ norm is of order p+1 (i.e. rate of $\mathcal{H}^1$ norm of u^h is p+1)
- A simple post-process (loop over elements) allows to compute a **superconvergent solution**

$$u_\star \in \mathcal{V}^h_\star(\Omega) := \{v \in \mathcal{L}_2(\Omega) : v|_{\Omega_e} \in \mathcal{P}^{\text{p}+1}(\Omega_e)\ \forall \Omega_e,\ e = 1,\dots,\texttt{n}_{\texttt{el}}\}$$

$$\begin{cases} -\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} u_\star = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_e & \text{in } \Omega_e, \\ \boldsymbol{n}_e\cdot\boldsymbol{\nabla} u_\star = -\boldsymbol{n}\cdot\boldsymbol{\sigma}_e & \text{on } \partial\Omega_e, \end{cases} \quad \& \quad \int_{\Omega_e} u_\star\, d\Omega = \int_{\Omega_e} u_e\, d\Omega$$

Constraint to eliminate indeterminacy

HDG convergence

- Using polynomials of degree p the following optimal rates of convergence are demonstrated for the $\mathcal{L}_2$ norm of

Variable	Order of convergence
u	p+1
$\hat{u}$	p+1
$\boldsymbol{\sigma}$	p+1
$u_\star$	p+2

Additional accuracy obtained by solving a set of ***small*** problems defined **independently element-by-element**

On accuracy and computational cost of HDG

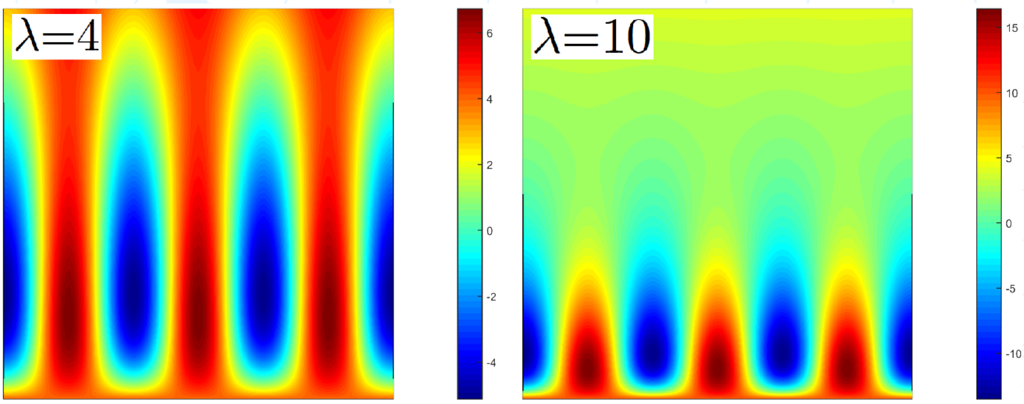
Example: Poisson equation

- Solve $\begin{cases} -\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} u = f & \text{in } \Omega, \\ u = u_D & \text{on } \Gamma_D, \\ \boldsymbol{n}\cdot\boldsymbol{\nabla} u = t & \text{on } \Gamma_N, \end{cases}$

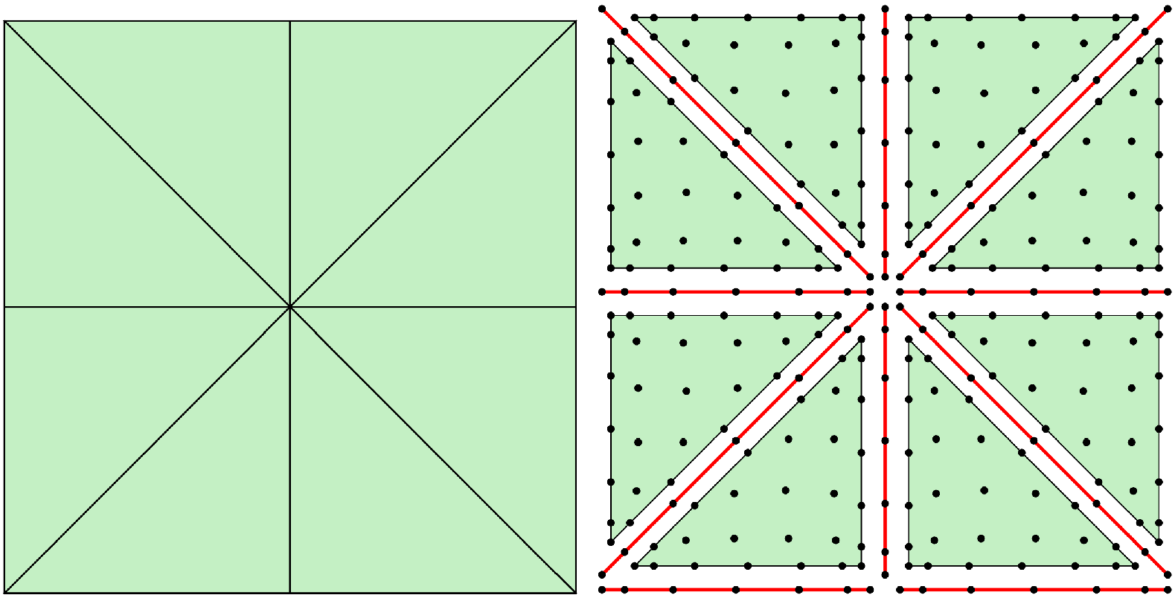
$$\Omega :=]0,1[\times]0,1[$$
$$\Gamma_N = \{(x,y)\in\partial\Omega \mid y=0\}$$
$$\Gamma_D = \partial\Omega\setminus\Gamma_N$$

with analytical solution:

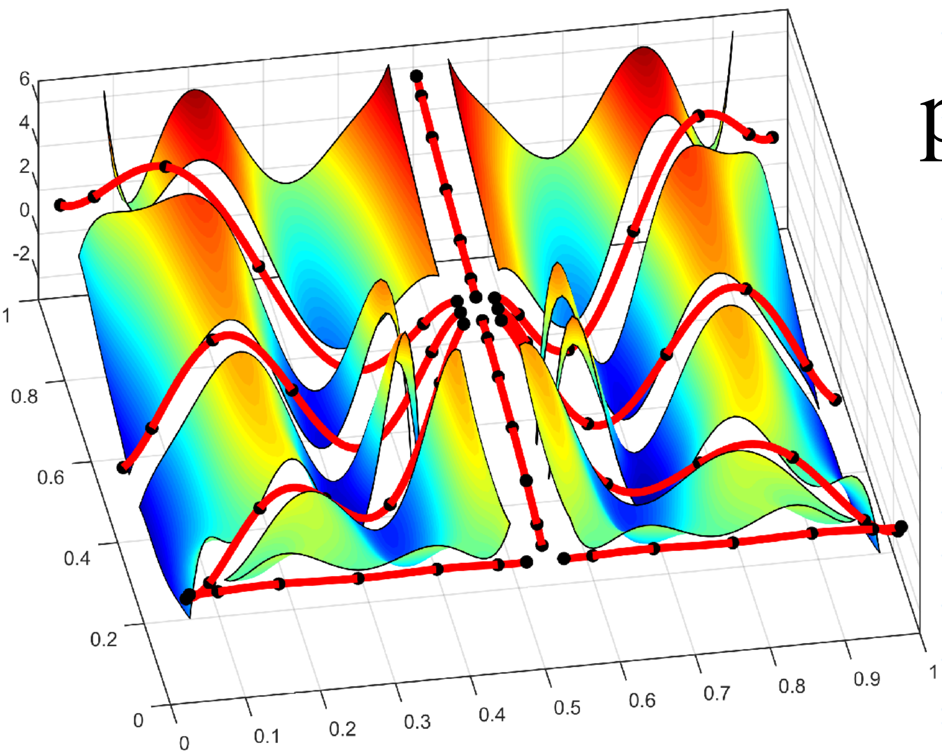
$$u(x,y) = 4y^2 - 4\lambda^2 y\exp(-\lambda y)\cos(6\pi x) + \lambda\exp(-2\lambda y)$$



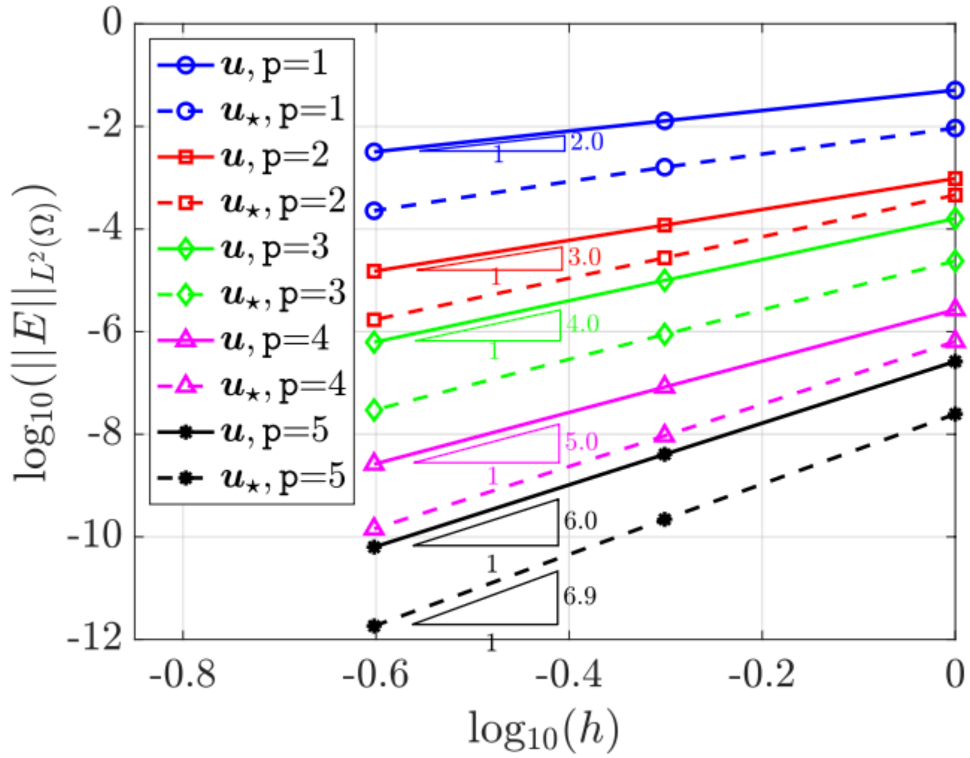
- Coarse mesh with p=6



$\lambda=4$



$p = 6$



High-order vs. Low-order

- [Huerta, Angeloski, Roca, Peraire, IJNME 2013] :
  "high-order elements are consistently more efficient than linear ones when confronted with the cost for solving the global system arising from second order differential operators.

  […]

  The comparison has been also extended to discontinuous versus continuous approaches for a given accuracy (i.e. mesh discretization) where it is also shown that the only method with a cost comparable to CG is HDG, provided its super-convergent property is utilized."

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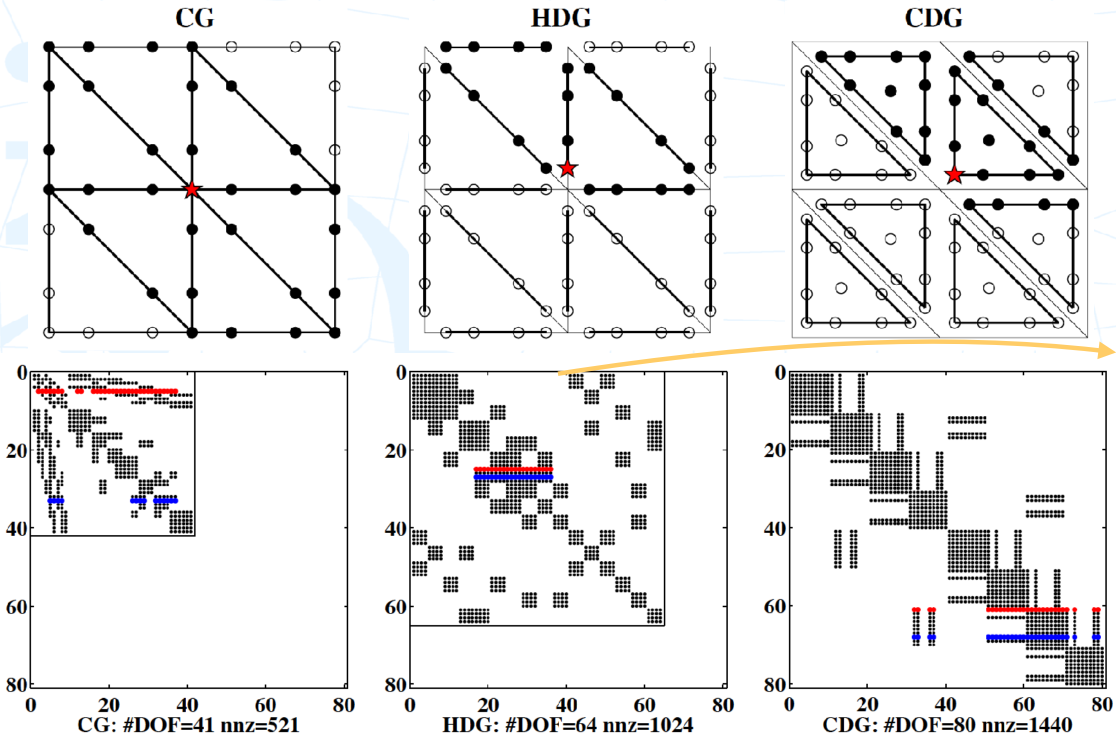
Stencils



All rows have the **same number of coefficients** and **same block structure**

Efficient linear solvers suitable for HDG

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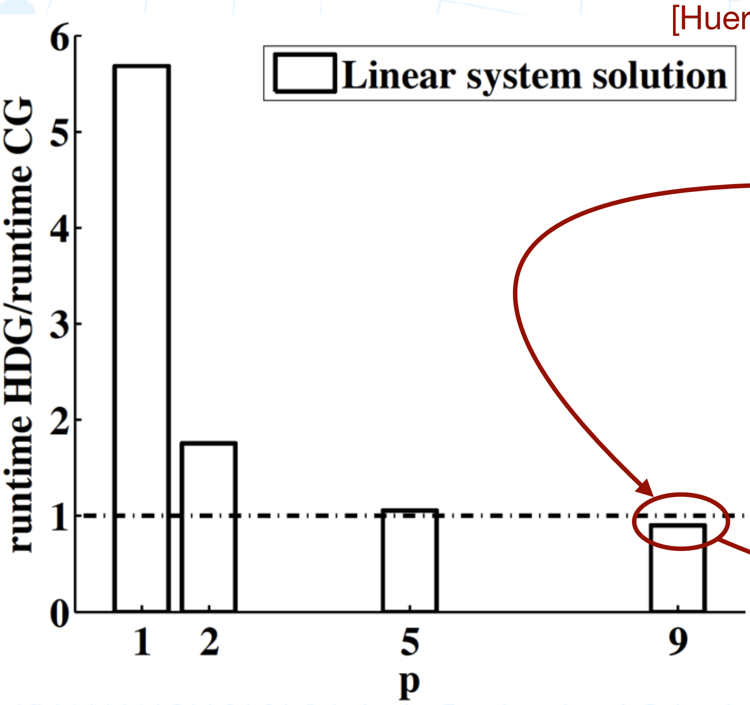
Solver cost: HDG vs. CG

[Huerta, Angeloski, Roca, Peraire. IJNME 2013]



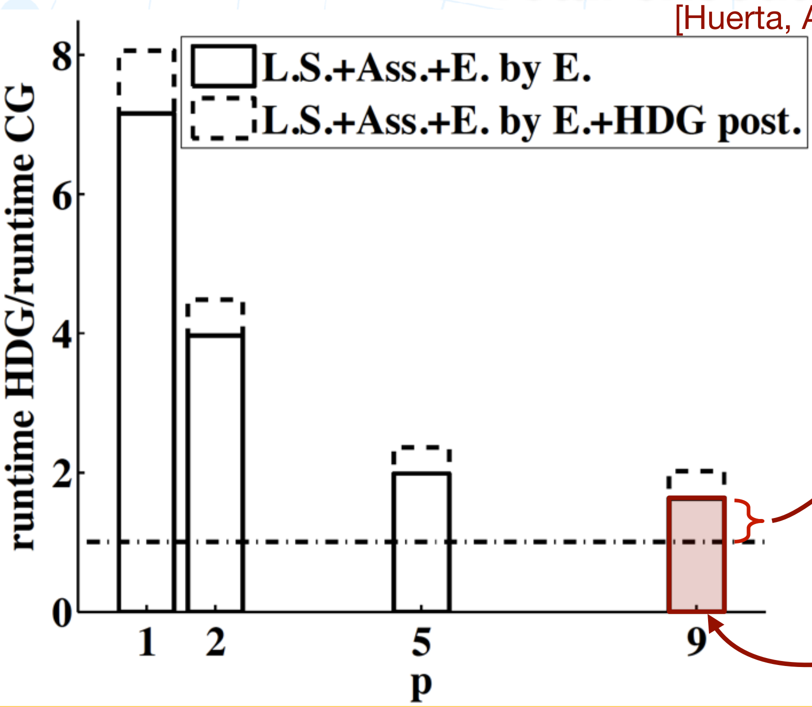
CG: 781 226 DOFs
➠ 5161 bandwidth

HDG: 937 230 DOFs
➠ 3438 bandwidth

HDG features:
20% **more** DOFs
33% **less** bandwidth
10% runtime reduction

Total execution time: HDG vs. CG

[Huerta, Angeloski, Roca, Peraire. IJNME 2013]



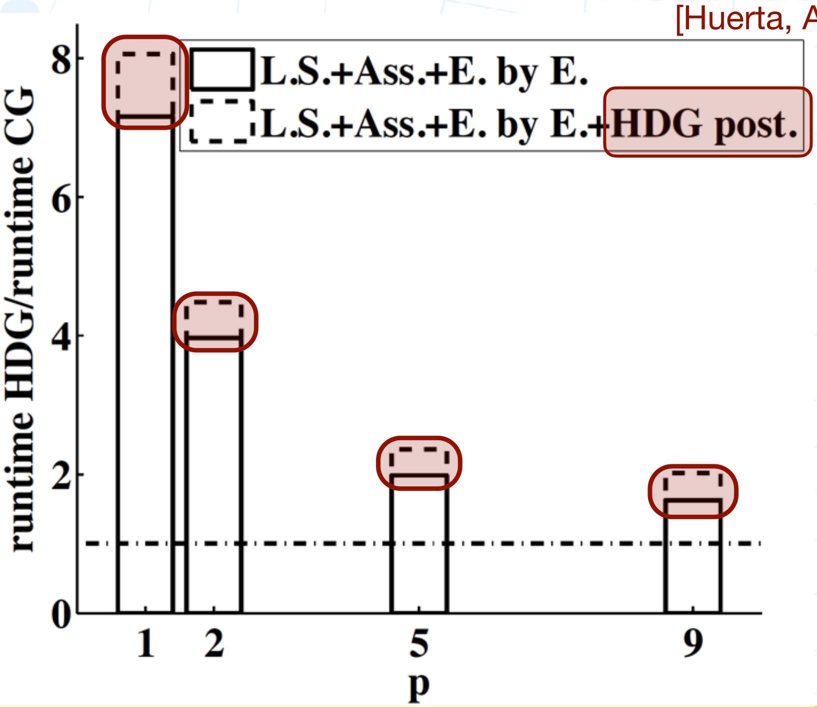
Total HDG runtime 60% more expensive

Accounting also for:

- assembly
- element-by-element computation

Accuracy and execution time

[Huerta, Angeloski, Roca, Peraire. IJNME 2013]



HDG postprocess allows to achieve **1 extra order of convergence** for a comparable computing effort

Summary I

- HDG outperforms DG, greatly reducing the number of DOFs.
- HDG has a slightly larger number of DOFs than CG.
- HDG provides stable approximations using the same order of polynomial discretisations for all the variables (easier mesh generation).
- HDG features a sparsity pattern particularly advantageous for the development of efficient linear solvers.