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# Domain Decomposition for Local Surrogate Models of Parametric Systems

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INTERNATIONAL CENTRE FOR NUMERICAL METHODS IN ENGINEERING
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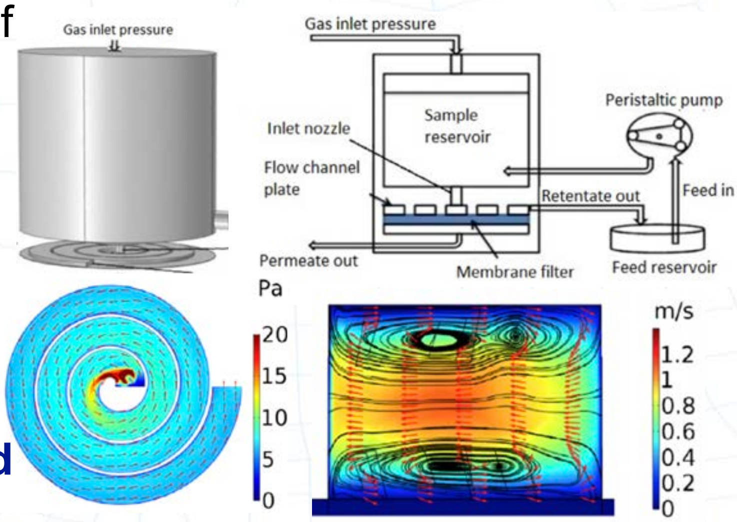
## Towards digital twins of water filtration systems…

**Goal:** Virtual prototyping of **membranes** for conception, design, optimisation, and operation of **micro-filtration** systems

6 CLEAN WATER AND SANITATION

Requirements for **digital twins** of water purification systems:

- Incorporate user-defined/ uncertain **parameters** (materials, geometry, functioning conditions, …).
- Encapsulate the underlying **physics** to be interpretable.
- Be compatible with **established solution** strategies in industry.

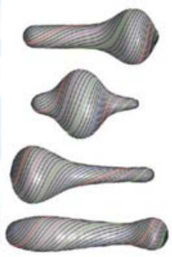



[Parasyris, Discacciati, Das. Appl Math Model 2020]

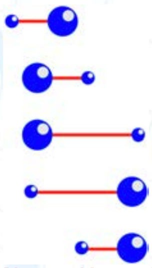
# Parametric design and optimisation

## **PMPY microswimmer:** simplified model of **euglenoid movement**

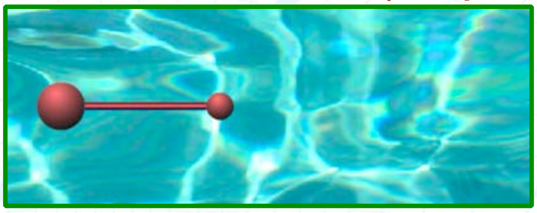
- Parametrised geometry

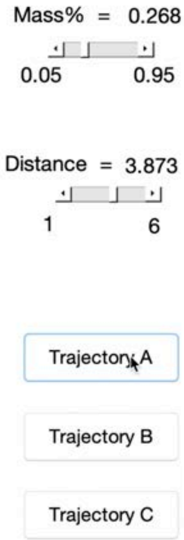


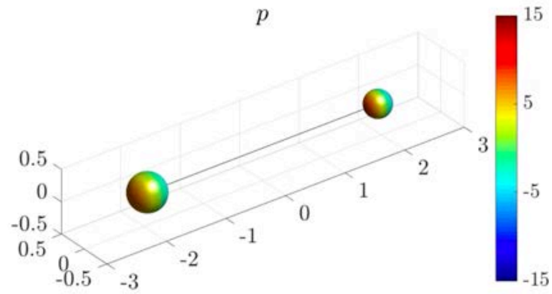
[Arroyo, Heltai, Millán, DeSimone. PNAS 2012]

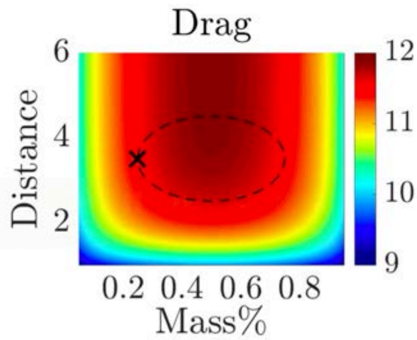


[Avron, Kenneth, Oaknin. New J Phys 2005]






[Sevilla, Borchini, Giacomini, Huerta. CMAME, 2020]
[Giacomini, Borchini, Sevilla, Huerta. Finite Elem Anal Des, 2021]

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# What are the computational bottlenecks?

- Multiple scenarios to be tested (many parameters)

→ **High dimensional problems**

- Large-scale systems
- Complex models (possibly, multi-physics)

→ **Demanding spatial solves**

**How to reduce the cost of constructing digital twins?**

- ➡ **Local surrogate models**
- ➡ Data augmentation
- ➡ Multi-fidelity approaches
- ➡ …

# Local Surrogate Models via Overlapping Domain Decomposition

with **Loughborough University**

Marco Discacciati

Ben Evans



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## Parametric multi-domain formulations

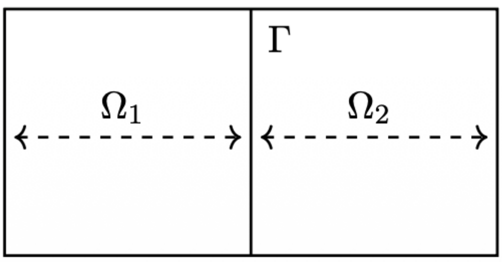
For each value of the parameter:

$$\begin{aligned} -\nabla \cdot (K(\mu)\nabla u(\mu)) &= s(\mu) \quad \text{in } \Omega \\ u(\mu) &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

→

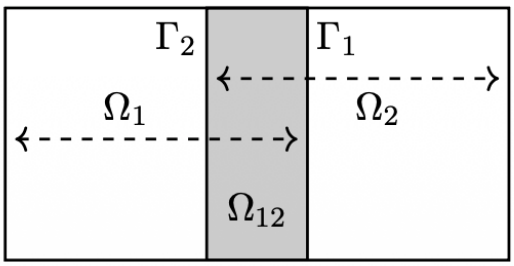
$$\begin{aligned} -\nabla \cdot (K_i(\mu)\nabla u_i(\mu)) &= s_i(\mu) \quad \text{in } \Omega_i \quad (i = 1, 2) \\ u_i(\mu) &= 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega \end{aligned}$$

### Non-overlapping




$$\begin{aligned} K_1(\mu)\nabla u_1(\mu)) \cdot \mathbf{n} & \\ = K_2(\mu)\nabla u_2(\mu)) \cdot \mathbf{n} & \quad \text{on } \Gamma \\ u_1(\mu) = u_2(\mu) & \quad \text{on } \Gamma \end{aligned}$$

### Overlapping




$u_1(\mu) = u_2(\mu)$ in $\Omega_{12}$

or

$u_1(\mu) = u_2(\mu)$ on $\Gamma_1 \cup \Gamma_2$

[Discacciati, Gervasio, Quarteroni. SICON 2013]

# Alternating Schwarz method

For each value of the parameter, solve:

$$\begin{pmatrix} A_{\Omega_1} & A_{\Gamma_1} & 0 & 0 \\ 0 & I_{\Gamma_1} & -R_{\Omega_2\to\Gamma_1} & 0 \\ 0 & 0 & A_{\Omega_2} & A_{\Gamma_2} \\ -R_{\Omega_1\to\Gamma_2} & 0 & 0 & I_{\Gamma_2} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{\Omega_1}(\bar\mu) \\ \mathbf{u}_{\Gamma_1}(\bar\mu) \\ \mathbf{u}_{\Omega_2}(\bar\mu) \\ \mathbf{u}_{\Gamma_2}(\bar\mu) \end{pmatrix} = \begin{pmatrix} \mathbf{s}_{\Omega_1}(\bar\mu) \\ \mathbf{0} \\ \mathbf{s}_{\Omega_2}(\bar\mu) \\ \mathbf{0} \end{pmatrix}$$

Schur complement leading to the interface system

$$\begin{pmatrix} I_{\Gamma_1} & R_{\Omega_2\to\Gamma_1} A_{\Omega_2}^{-1} A_{\Gamma_2} \\ R_{\Omega_1\to\Gamma_2} A_{\Omega_1}^{-1} A_{\Gamma_1} & I_{\Gamma_2} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{\Gamma_1}(\bar\mu) \\ \mathbf{u}_{\Gamma_2}(\bar\mu) \end{pmatrix} = \begin{pmatrix} R_{\Omega_2\to\Gamma_1} A_{\Omega_2}^{-1} \mathbf{s}_{\Omega_2}(\bar\mu) \\ R_{\Omega_1\to\Gamma_2} A_{\Omega_1}^{-1} \mathbf{s}_{\Omega_1}(\bar\mu) \end{pmatrix}$$

Solution of local FEM problems can become unaffordable in a parametric setting!





# Surrogate-based alternating Schwarz method

For each value of the parameter, solve:

$$\begin{pmatrix} A_{\Omega_1} & A_{\Gamma_1} & 0 & 0 \\ 0 & I_{\Gamma_1} & -R_{\Omega_2\to\Gamma_1} & 0 \\ 0 & 0 & A_{\Omega_2} & A_{\Gamma_2} \\ -R_{\Omega_1\to\Gamma_2} & 0 & 0 & I_{\Gamma_2} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{\Omega_1}(\bar\mu) \\ \mathbf{u}_{\Gamma_1}(\bar\mu) \\ \mathbf{u}_{\Omega_2}(\bar\mu) \\ \mathbf{u}_{\Gamma_2}(\bar\mu) \end{pmatrix} = \begin{pmatrix} \mathbf{s}_{\Omega_1}(\bar\mu) \\ \mathbf{0} \\ \mathbf{s}_{\Omega_2}(\bar\mu) \\ \mathbf{0} \end{pmatrix}$$

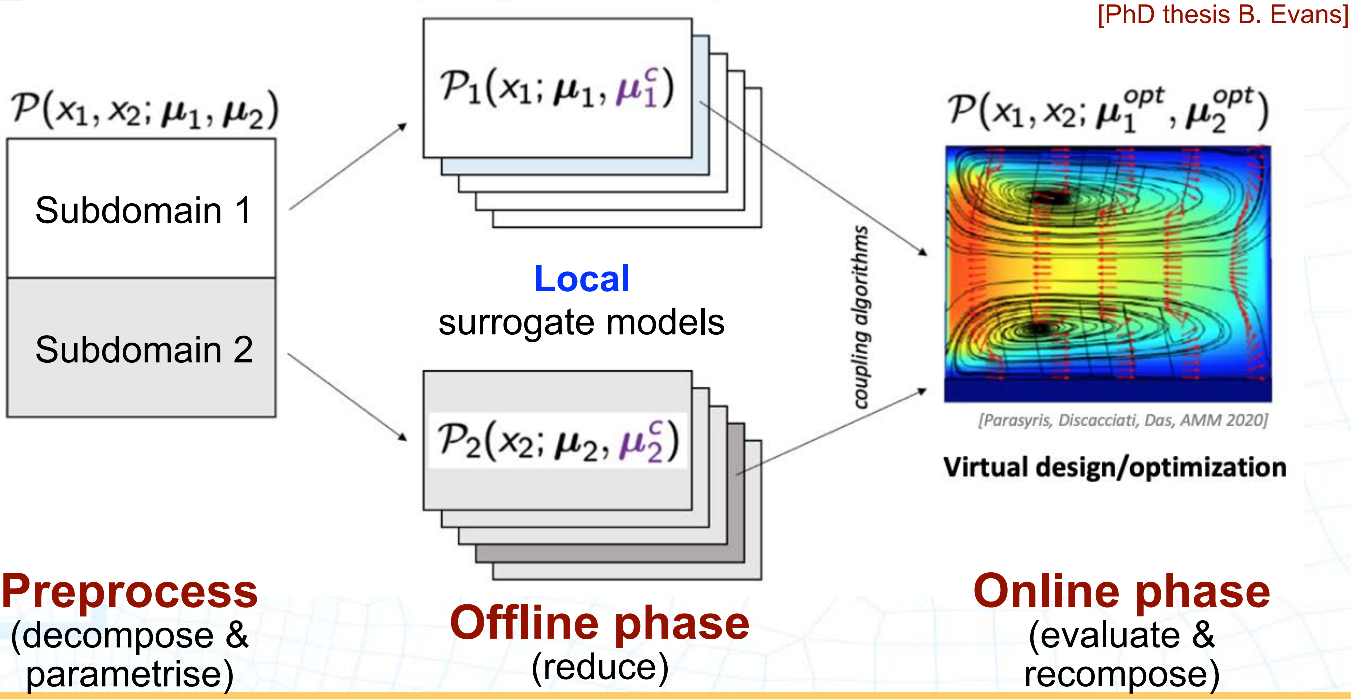
Schur complement leading to the interface system

$$\begin{pmatrix} I_{\Gamma_1} & R_{\Omega_2\to\Gamma_1} A_{\Omega_2}^{\text{PGD}} \\ R_{\Omega_1\to\Gamma_2} A_{\Omega_1}^{\text{PGD}} & I_{\Gamma_2} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{pmatrix} = \begin{pmatrix} R_{\Omega_2\to\Gamma_1} \mathbf{u}_{2,\text{PGD}}^{s}(\bar\mu) \\ R_{\Omega_1\to\Gamma_2} \mathbf{u}_{1,\text{PGD}}^{s}(\bar\mu) \end{pmatrix}$$

Replace by means of a previously surrogate model **(inexpensive evaluation!)**

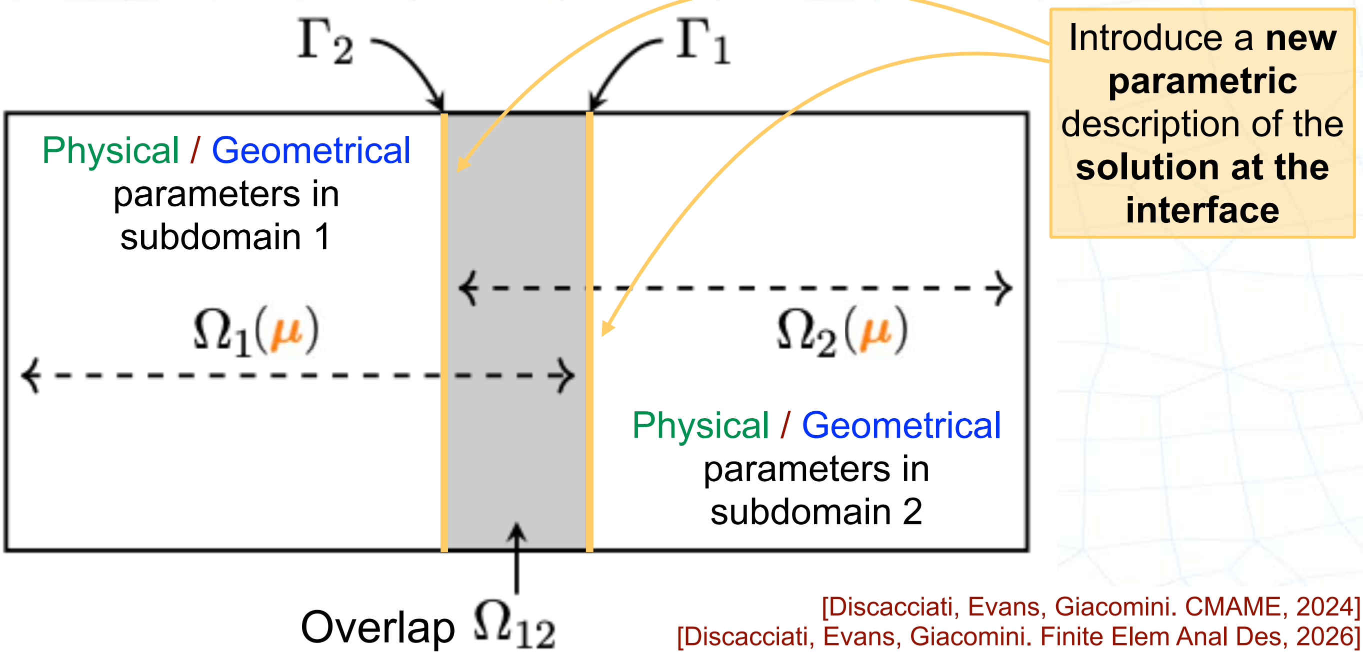
# Decompose > Reduce > Recompose

[PhD thesis B. Evans]






# Preprocess: decompose into parametric subdomains




[Discacciati, Evans, Giacomini. CMAME, 2024]
[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

# Offline: define a parametric subproblem

In each subdomain, solve the **linear** elliptic PDE:

$$
\begin{aligned}
L(u_i(\boldsymbol{\mu});\boldsymbol{\mu}) &= s_i(\boldsymbol{\mu}) && \text{in } \Omega_i\,,\\
u_i(\boldsymbol{\mu}) &= g_i^D(\boldsymbol{\mu}) && \text{on } \Gamma_i^D,\\
\nu(\boldsymbol{\mu})\nabla u_i(\boldsymbol{\mu})\cdot \mathbf{n} &= g_i^N(\boldsymbol{\mu}) && \text{on } \Gamma_i^N,\\
u_i(\boldsymbol{\mu}) &= \lambda_i && \text{on } \Gamma_i\,,
\end{aligned}
$$

$$\lambda_i = \lambda_i(\mathbf{x}) = \sum_{q=1}^{N_{\Gamma_i}} \Lambda_i^q \eta_i^q(\mathbf{x})$$

**User-defined** spatial functions (**FEM basis**)

New set of **parameters** (To be determined **online**)

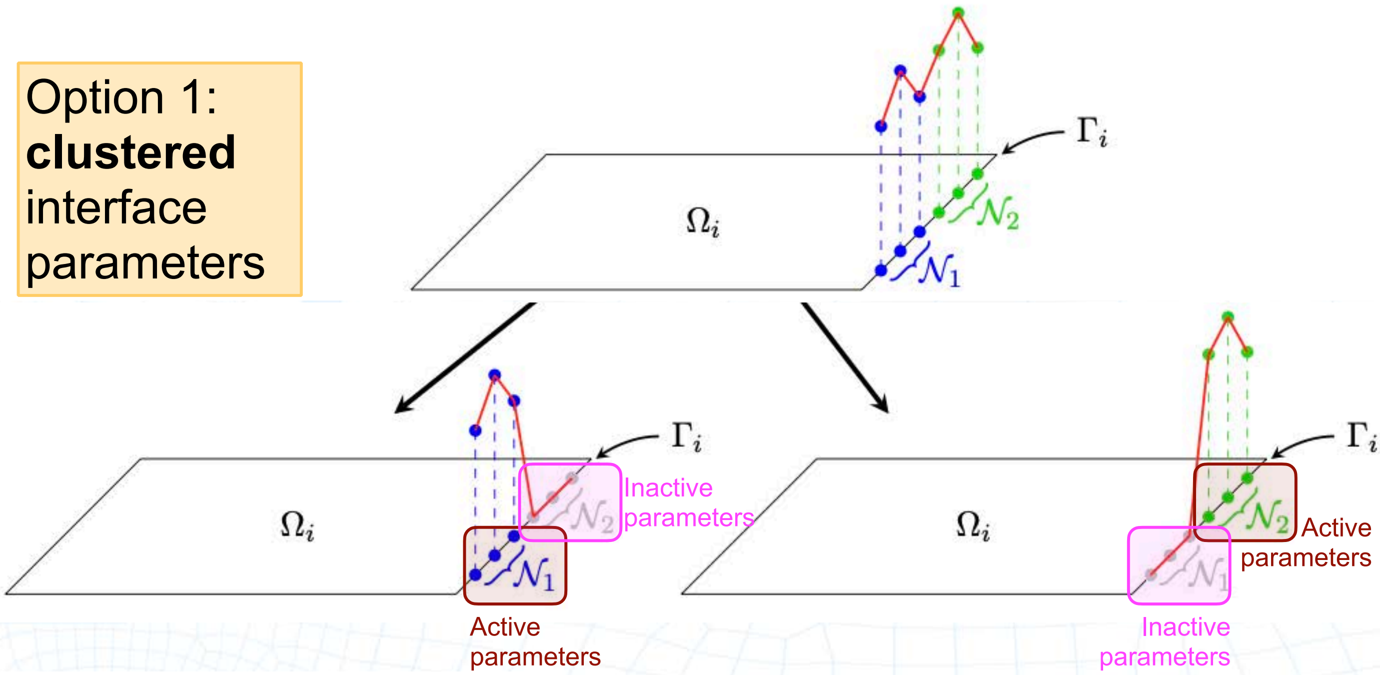
Less DOFs in space but **higher dimensional** problem!

[Discacciati, Evans, Giacomini. CMAME, 2024]
[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]



# Offline: reduce the dimensionality

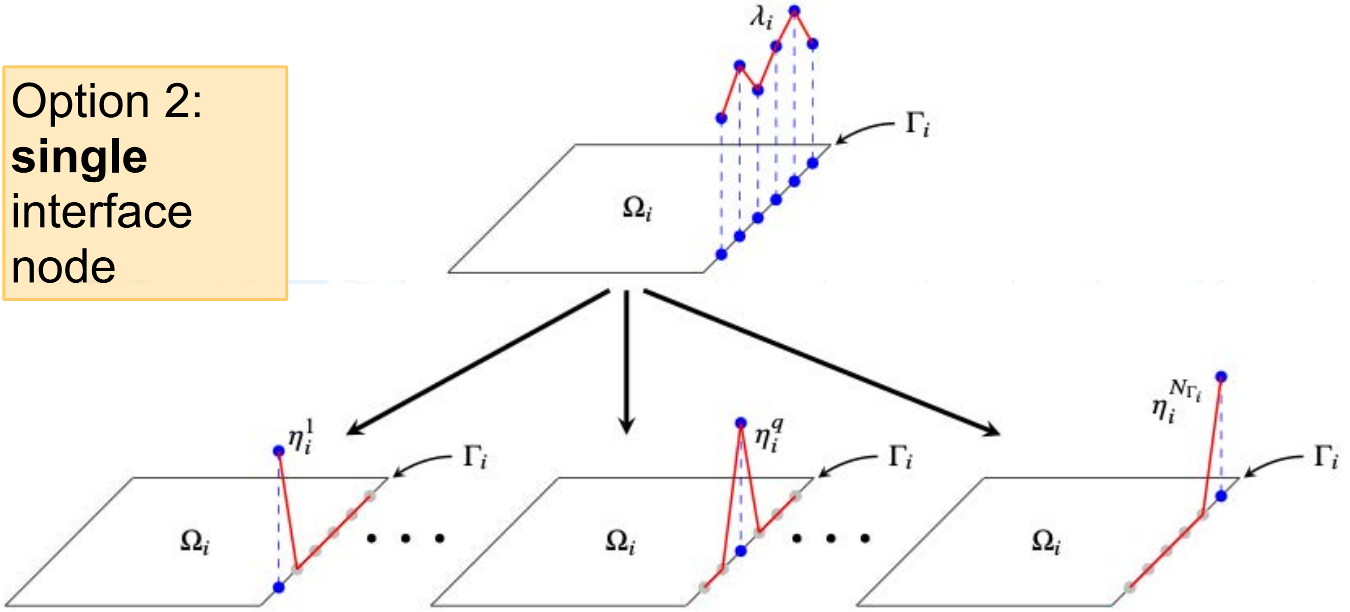
Option 1: **clustered** interface parameters




[Discacciati, Evans, Giacomini. CMAME, 2024]

# Offline: reduce the dimensionality

Option 2:
**single**
interface
node




[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]



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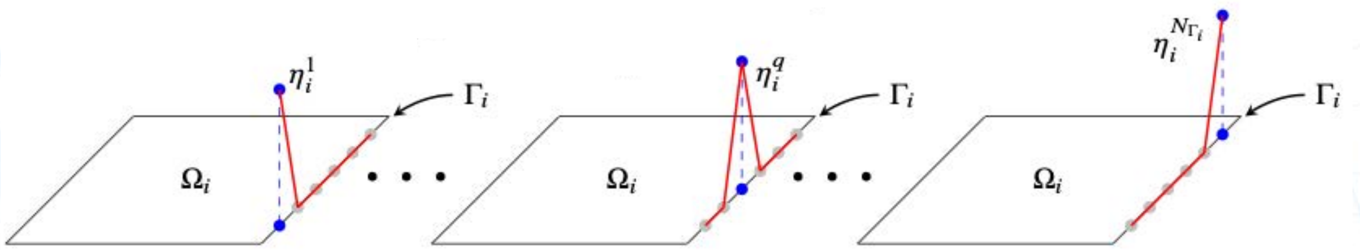
# Offline: construct a PGD local surrogate model

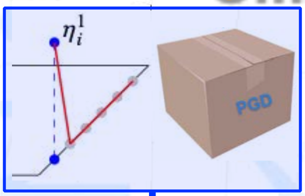
In each subdomain, construct $N_{\Gamma_i}$ local surrogate models, each associated with the non-zero BC on one node of the interface.




$$\boldsymbol{u}_{i,j}(\boldsymbol{x},\boldsymbol{\mu}) \simeq \boldsymbol{u}^{\texttt{PGD}}_{i,j}(\boldsymbol{x},\boldsymbol{\mu}) = \sum_{m=1}^{n} \sigma_u^m \boldsymbol{f}_u^m(\boldsymbol{x})\, \phi^m(\boldsymbol{\mu}) \qquad j=1,\dots,N_{\Gamma_i}$$

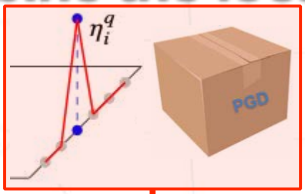
[Discacciati, Evans, Giacomini. CMAME, 2024]
[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

# Offline: combine the local surrogate models



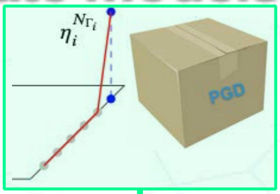

$\boldsymbol{u}_{i,1}^{\text{PGD}}$

…




$\boldsymbol{u}_{i,q}^{\text{PGD}}$

…




$\boldsymbol{u}_{i,N_{\Gamma_i}}^{\text{PGD}}$

Retrieve the surrogate model for subdomain $\Omega_i$

$$\boldsymbol{u}_i^{\text{PGD}}(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\Lambda}_i) = \sum_{q=1}^{N_{\Gamma_i}} \Lambda_i^q \boldsymbol{u}_{i,q}^{\text{PGD}}(\boldsymbol{x}, \boldsymbol{\mu})$$

Because of linearity of the PDE!

[Discacciati, Evans, Giacomini. CMAME, 2024]
[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]



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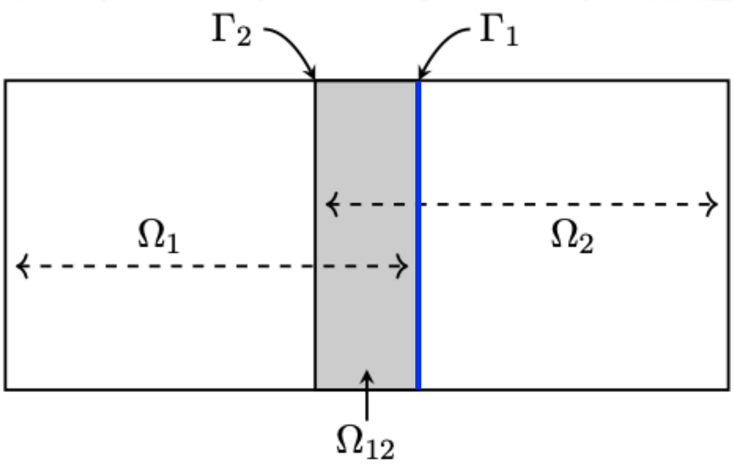
# Online: parametric overlapping Schwarz

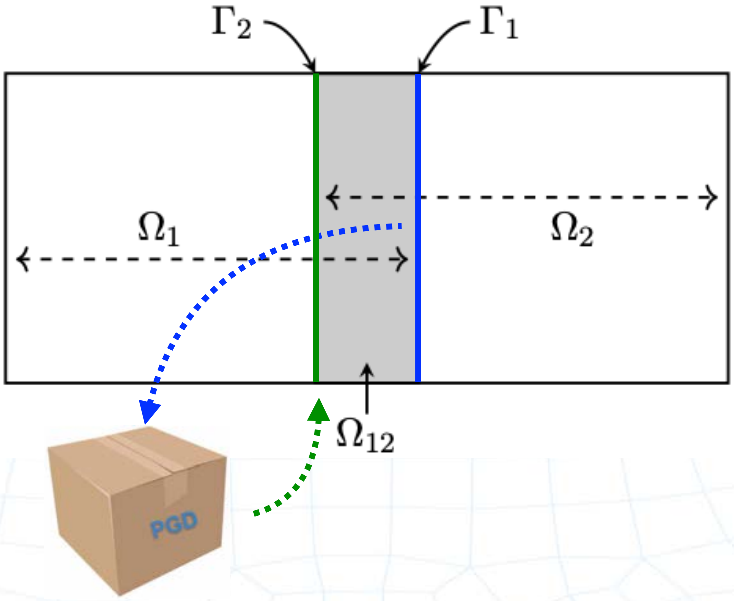
1. Fix a set of boundary parameters $\Lambda_1$ describing the solution on $\Gamma_1$




[Discacciati, Evans, Giacomini. CMAME, 2024]
[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

# Online: parametric overlapping Schwarz

1. Fix a set of boundary parameters $\Lambda_1$ describing the solution on $\Gamma_1$
2. Evaluate the PGD solution $\boldsymbol{u}_1^{\mathrm{PGD}}$ on $\Gamma_2$

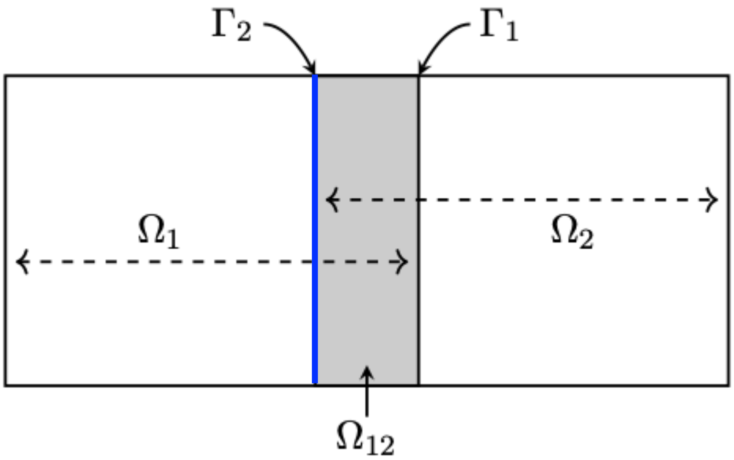



[Discacciati, Evans, Giacomini. CMAME, 2024]
[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]



# Online: parametric overlapping Schwarz

1. Fix a set of boundary parameters $\Lambda_1$ describing the solution on $\Gamma_1$
2. Evaluate the PGD solution $\boldsymbol{u}_1^{\mathrm{PGD}}$ on $\Gamma_2$
3. Extract the boundary parameters $\Lambda_2$ on $\Gamma_2$




[Discacciati, Evans, Giacomini. CMAME, 2024]
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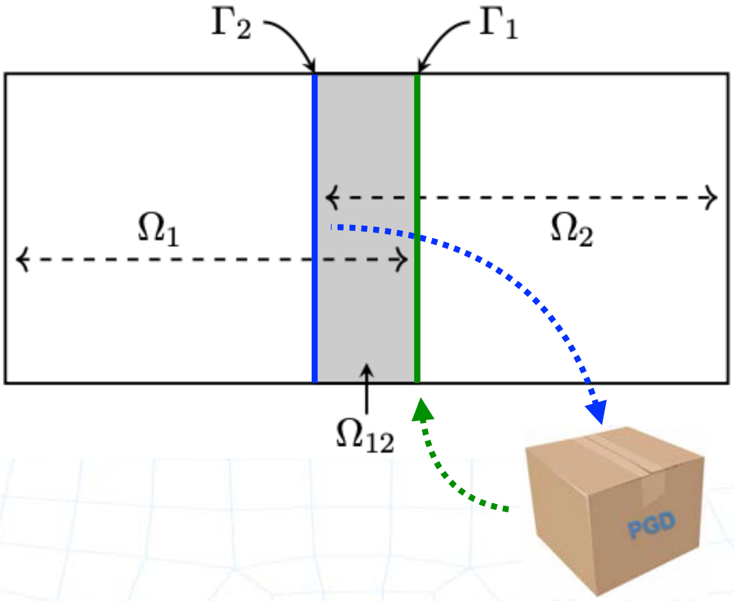
# Online: parametric overlapping Schwarz

1. Fix a set of boundary parameters $\Lambda_1$ describing the solution on $\Gamma_1$
2. Evaluate the PGD solution $\boldsymbol{u}_1^{\text{PGD}}$ on $\Gamma_2$
3. Extract the boundary parameters $\Lambda_2$ on $\Gamma_2$
4. Evaluate the PGD solution $\boldsymbol{u}_2^{\text{PGD}}$ on $\Gamma_1$

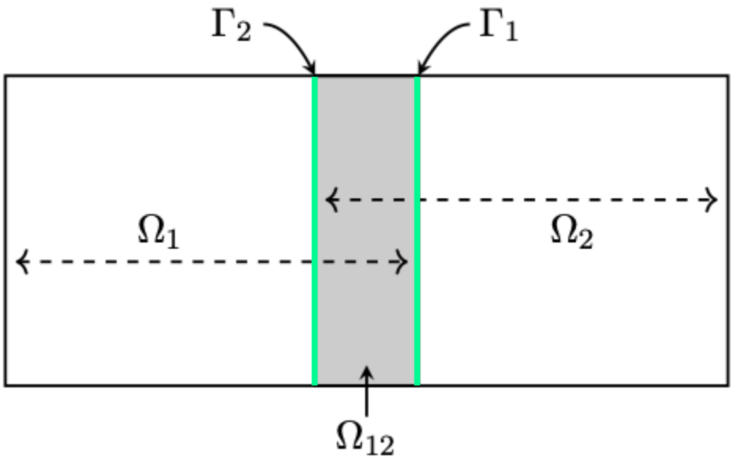



[Discacciati, Evans, Giacomini. CMAME, 2024]
[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]



# Online: parametric overlapping Schwarz

1. Fix a set of boundary parameters $\Lambda_1$ describing the solution on $\Gamma_1$
2. Evaluate the PGD solution $\boldsymbol{u}_1^{\text{PGD}}$ on $\Gamma_2$
3. Extract the boundary parameters $\Lambda_2$ on $\Gamma_2$
4. Evaluate the PGD solution $\boldsymbol{u}_2^{\text{PGD}}$ on $\Gamma_1$
5. Check convergence and go to 1.




[Discacciati, Evans, Giacomini. CMAME, 2024]
[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

# Encapsulated Proper Generalised Decomposition

[Ammar, Mokdad, Chinesta, Keunings. J Nonnewton Fluid Mech 2006]
[Díez, Zlotnik, García-González, Huerta. ACME, 2020]

- The commercial/industrial software selected by the user for the spatial problem is used to preprocess data for the construction of the discrete high-dimensional parametric problem.

$$\boldsymbol{u}^{\mathrm{PGD}}_{i,j}(\boldsymbol{x},\boldsymbol{\mu}) = \sum_{m=1}^{n} \sigma_u^m \boldsymbol{f}_u^m(\boldsymbol{x})\, \phi^m(\boldsymbol{\mu}) = \sum_{m=1}^{n} \sigma_u^m \boldsymbol{f}_u^m(\boldsymbol{x}) \prod_{k=1}^{\mathtt{n_{pa}}} \phi_k^m(\mu_k)$$

Discretised using finite elements, finite volumes, … according to the format of the **selected spatial solver**.

Discretised using **pointwise collocation**.

- Vectors and matrices of the spatial component of the high-dimensional problem are extracted from the spatial solver.

$$\mathbf{K}(\boldsymbol{x},\boldsymbol{\mu})\mathbf{u}(\boldsymbol{x},\boldsymbol{\mu}) = \mathbf{b}(\boldsymbol{x},\boldsymbol{\mu})$$



# Encapsulated Proper Generalised Decomposition

[Díez, Zlotnik, García-González, Huerta. ACME, 2020]

**Parametric system**

$$\mathbf{K}(\boldsymbol{x},\boldsymbol{\mu})\mathbf{u}(\boldsymbol{x},\boldsymbol{\mu}) = \mathbf{b}(\boldsymbol{x},\boldsymbol{\mu})$$

$\boldsymbol{\mu} = [\mu_1, \dots, \mu_{\mathtt{n_{pa}}}]^\top$

**Black-box software**

$\mathbf{K}(\boldsymbol{x},\boldsymbol{\mu})$
$\mathbf{b}(\boldsymbol{x},\boldsymbol{\mu})$

Matrices and vectors in separated form

**Preprocessing stage**

ePGD

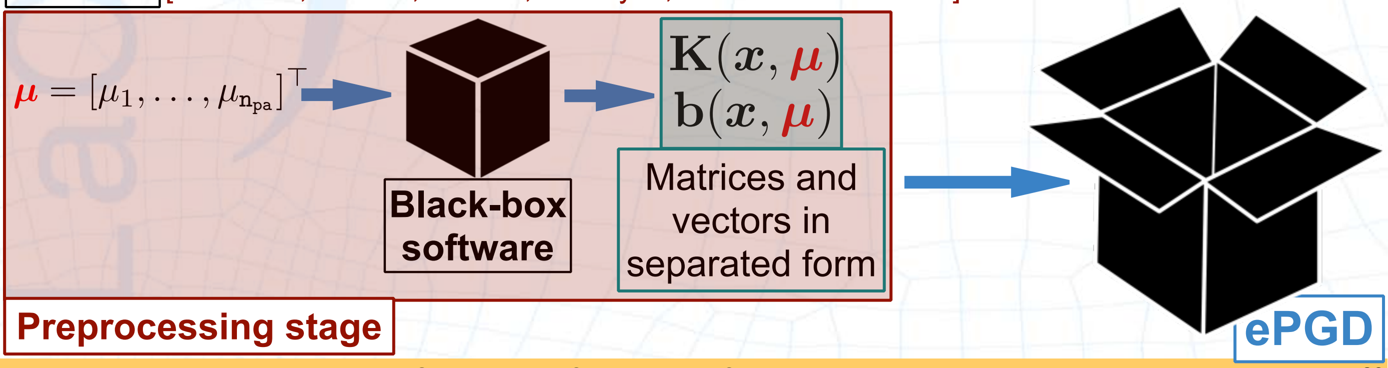
# Encapsulated Proper Generalised Decomposition

[Díez, Zlotnik, García-González, Huerta. ACME, 2020]

- ePGD offers a seamless implementation of PGD for the end user (invisible greedy and ADI routines).
- Suitable for commercial and industrial environment.
- Generic implementation independent of the spatial solver.

**VPS/Pam-Crash** [Rocas, García-González, Zlotnik, Larráyoz, Díez. FINEL 2021]

**NASTRAN** [Cavaliere, Zlotnik, Sevilla, Larrayoz, Díez. CMAME 2022]






# The notion of separated tensor

## Parametric model problem

$$-\nabla \cdot ((1 + \mu x)\nabla u) = s(\mu), \quad \text{in } \Omega \qquad \Omega = (0,2) \times (0,1) \qquad \mathcal{I} = [1, 50]$$

$$u = 0, \quad \text{on } \partial\Omega \qquad s(\mu) = b_s^1(\boldsymbol{x}) + \mu b_s^2(\boldsymbol{x}) + \mu^2 b_s^3(\boldsymbol{x})$$

## Parametric weak form

**Parametric modes**: vectors discretised in $\mathcal{I}$ with pointwise collocation

$$\underbrace{1}_{\boldsymbol{\xi}_\nu^1} \underbrace{\int_\Omega \nabla v \cdot \nabla u \, d\Omega}_{\mathbf{K}^1} + \underbrace{\mu}_{\boldsymbol{\xi}_\nu^2} \underbrace{\int_\Omega x \nabla v \cdot \nabla u \, d\Omega}_{\mathbf{K}^2} = \underbrace{1}_{\boldsymbol{\xi}_s^1} \underbrace{\int_\Omega v \, b_s^1(\boldsymbol{x}) \, d\Omega}_{\mathbf{b}^1} + \underbrace{\mu}_{\boldsymbol{\xi}_s^2} \underbrace{\int_\Omega v \, b_s^2(\boldsymbol{x}) \, d\Omega}_{\mathbf{b}^2} + \underbrace{\mu^2}_{\boldsymbol{\xi}_s^3} \underbrace{\int_\Omega v \, b_s^3(\boldsymbol{x}) \, d\Omega}_{\mathbf{b}^3}$$

**Spatial modes**: matrices and vectors discretised in $\Omega$ with black-box solver

**Separated tensor** $\mathbf{K}(\boldsymbol{x}, \boldsymbol{\mu})$

**Separated tensor** $\mathbf{b}(\boldsymbol{x}, \boldsymbol{\mu})$

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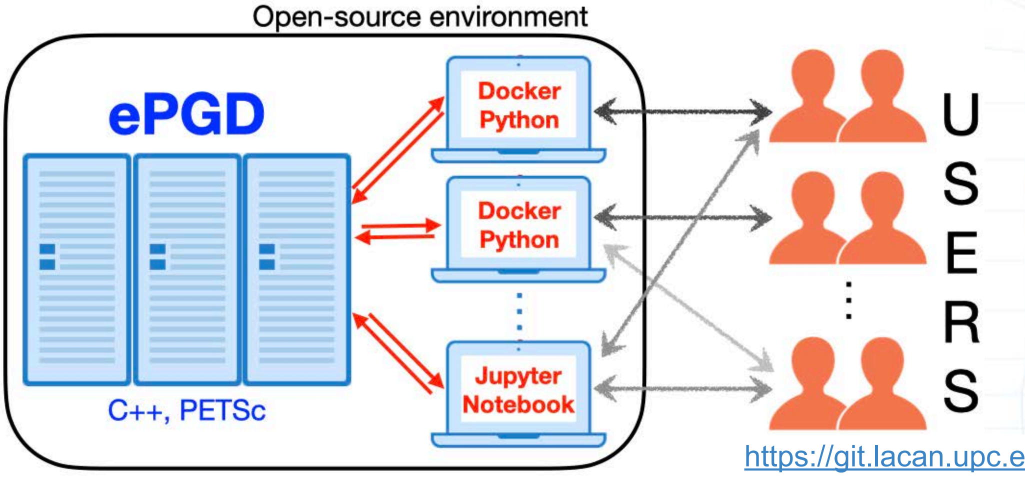
Prueba de Concepto PDC2021-121554-C21

# The ePGD library

- **Open-source** environment.
- **Efficient** core routines implemented in **C++**, linked with **PETSc**.
- **User-friendly** interface in **Python**.




ePGD demo

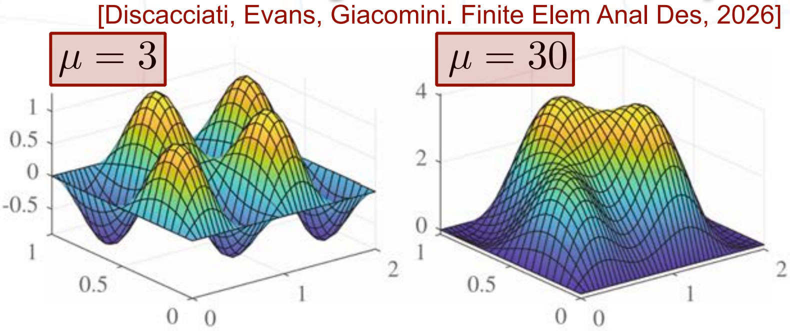
https://git.lacan.upc.edu/encapsulated-pgd/demo



# A synthetic example

[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

$$\begin{aligned} -\nabla \cdot ((1+\mu x)\nabla u(\mu)) &= s(\mu) \quad \text{in } \Omega\,, \\ u(\mu) &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$




- **Domain:** $\Omega = (0,2) \times (0,1)$
- **Decomposition:** $\Omega_1 = (0, 1 + nh_x) \times (0,1)$, $\Omega_2 = (1 - nh_x, 2) \times (0,1)$
- **Parametric domain:** $\mu \in \mathcal{P} = [1,50]$
- **Discretisation:** $\mathbb{Q}_1$ finite elements

Overlap: $2nh_x$

$h_\mu = h_\Lambda = 10^{-3}$
$h_x = 5 \times 10^{-2}$

[Discacciati, Evans, Giacomini. CMAME, 2024]
[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

# A synthetic example

[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

**Problem setup:**

- 2 spatial dimensions
- 1 parameter $\mu$
- 19 parametric nodes at the interface

| | Number of clustered interface parameters | Number of subproblems | Dimension of subproblems | Offline time (s) |
|---|---|---|---|---|
| **Single interface node** | - | 19 | 3 | 5.66 |
| **Clustered interface parameters** | 1 | 19 | 4 | 9.80 |
| | 3 | 7 | 6 | 112.14 |
| | 5 | 4 | 8 | 482.68 |



# A synthetic example

[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

**Problem setup:**

- 2 spatial dimensions
- 1 parameter $\mu$
- 19 parametric nodes at the interface

Space of the solution of dimension 22

| | Number of clustered interface parameters | Number of subproblems | **Dimension of subproblems** | Offline time (s) |
|---|---|---|---|---|
| **Single interface node** | - | 19 | 3 | 5.66 |
| **Clustered interface parameters** | 1 | 19 | 4 | 9.80 |
| | 3 | 7 | 6 | 112.14 |
| | 5 | 4 | 8 | 482.68 |

# A synthetic example

[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

## Problem setup:

- 2 spatial dimensions
- 1 parameter $\mu$
- 19 parametric nodes at the interface

**Accuracy comparable in all cases**

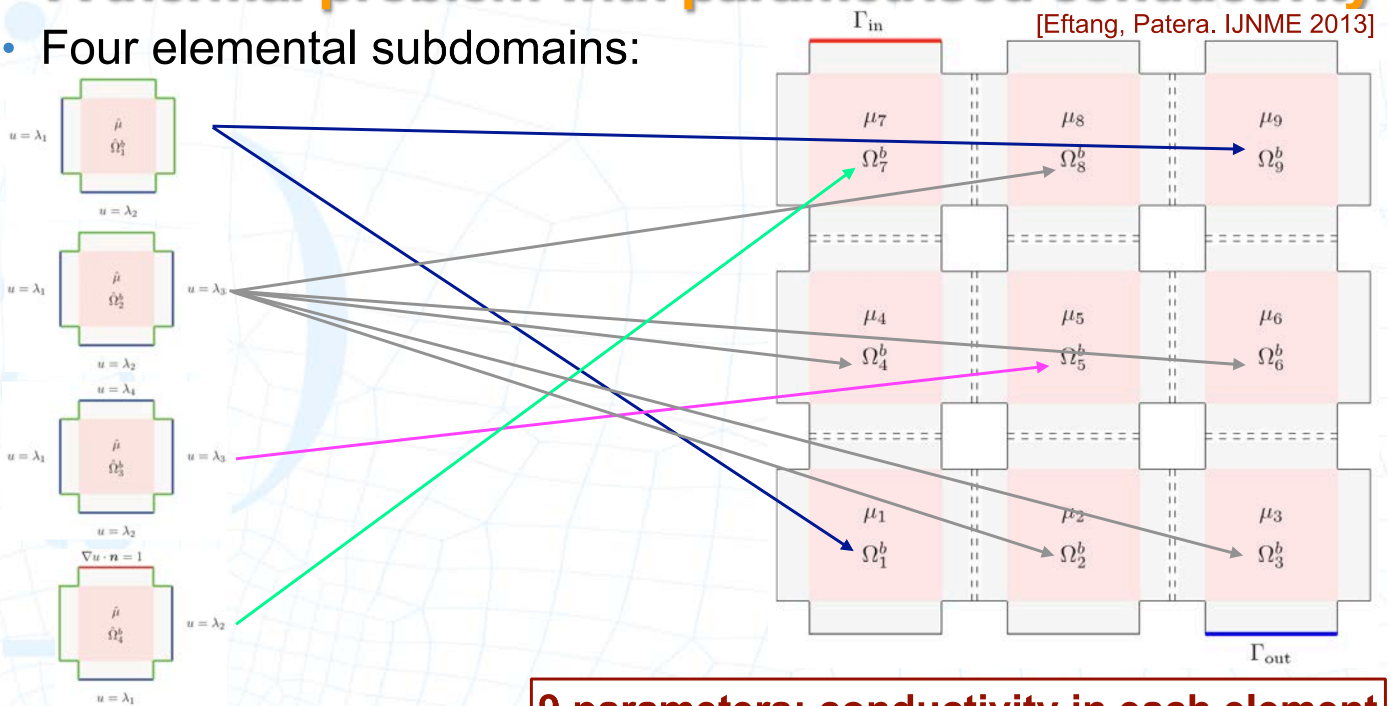
| | Number of clustered interface parameters | Number of subproblems | Dimension of subproblems | Offline time (s) | Number of GMRES iterations **(online)** |
|---|---|---|---|---|---|
| **Single interface node** | - | 19 | 3 | 5.66 | 9 |
| **Clustered interface parameters** | 1 | 19 | 4 | 9.80 | 13 |
| | 3 | 7 | 6 | 112.14 | 16 |
| | 5 | 4 | 8 | 482.68 | 18 |



# A thermal problem with parametrised conductivity

[Eftang, Patera. IJNME 2013]

- Four elemental subdomains:

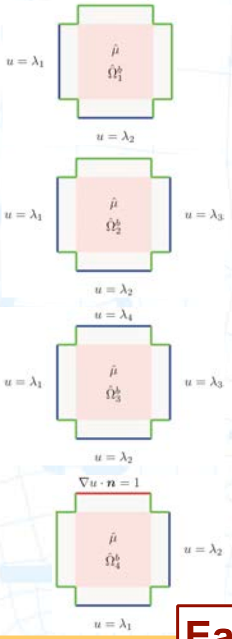



**9 parameters: conductivity in each element**

# A thermal problem with parametrised conductivity

[Eftang, Patera. IJNME 2013]

- Four elemental subdomains:

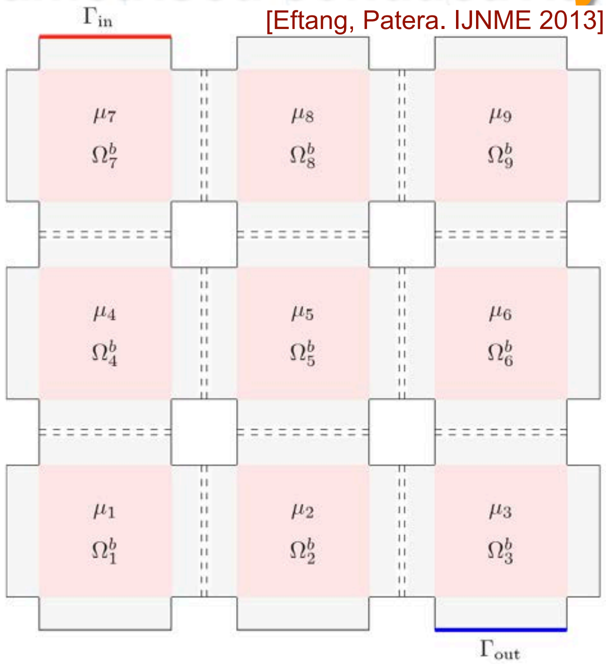


42 subproblems with 2,163 DOFs

63 subproblems with 2,142 DOFs

84 subproblems with 2,121 DOFs

43 subproblems with 2,163 DOFs

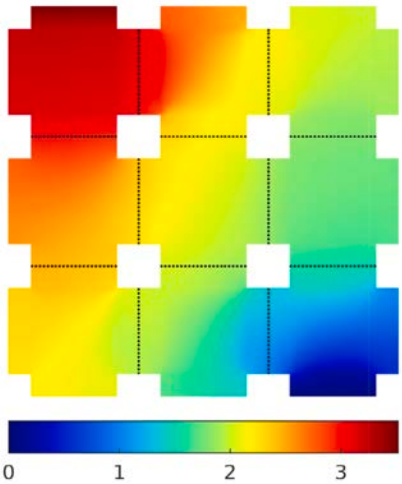



**Each subproblem with 2 spatial and 1 parametric dimensions**



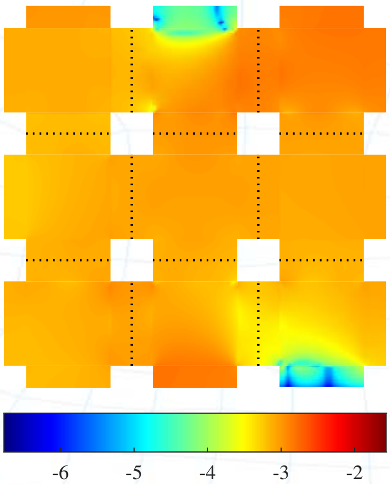
# DD-PGD accuracy

$u^{\mathrm{PGD}}(\boldsymbol{\mu})$

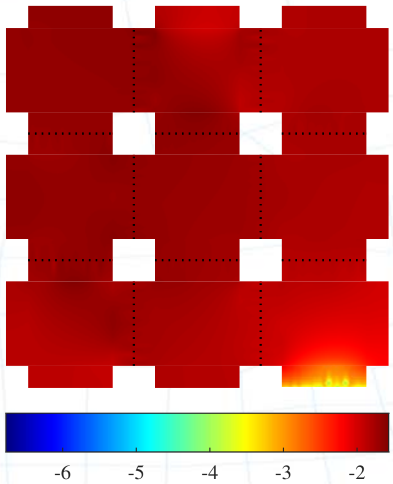



Temperature field

$\log_{10}(|u_{\mathrm{PGD}}(\hat{\mu}) - u^h(\hat{\mu})| / \max_\Omega |u^h(\hat{\mu})|)$




Single interface node: max error $1.8 \times 10^{-3}$




Clustered interface parameters: max error $2.2 \times 10^{-2}$

[Discacciati, Evans, Giacomini. CMAME, 2024]
[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

# DD-PGD efficiency

| | Number of clustered interface parameters | Number of subproblems | Dimension of subproblems | **Offline time (s)** | Number of GMRES iterations | **Online time (s)** |
|---|---|---|---|---|---|---|
| **Single interface node** | - | 231 | 3 | 36.18 | 93 | 0.47 |
| **Clustered interface parameters** | 3 | 77 | 6 | 872.38 | 302 | 157.23 |
| **Reference DD-FEM** | | | | | 95 | 414.80 |

**DD-PGD** speed-up of
**882 times** w.r.t. DD-FEM!

[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]



# DD-PGD efficiency


| | Number of clustered interface parameters | Number of subproblems | Dimension of subproblems | **Offline time (s)** | Number of GMRES iterations | **Online time (s)** |
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| **Reference DD-FEM** | | | | | 95 | 414.80 |

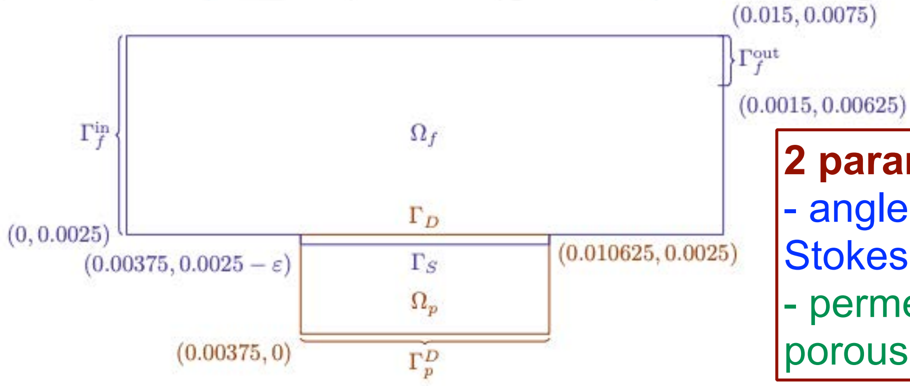

**DD-PGD** speed-up of
**11 times** w.r.t. DD-FEM!

[Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]

# The coupled multi-physics case

Fluid domain

Porous medium




**2 parameters:**
- angle of inlet Stokes velocity
- permeability of porous medium

### Stokes equations:

$$
\begin{aligned}
-\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_f(\boldsymbol{\mu}),p_f(\boldsymbol{\mu});\boldsymbol{\mu}) &= \mathbf{0} && \text{in } \Omega_f,\\
\nabla\cdot\boldsymbol{u}_f(\boldsymbol{\mu}) &= 0 && \text{in } \Omega_f,\\
\boldsymbol{u}_f(\boldsymbol{\mu}) &= \boldsymbol{g}_f^D(\boldsymbol{\mu}) && \text{on } \Gamma_f^{\text{in}},\\
\boldsymbol{\sigma}(\boldsymbol{u}_f(\boldsymbol{\mu}),p_f(\boldsymbol{\mu});\boldsymbol{\mu})\cdot\boldsymbol{n} &= \mathbf{0} && \text{on } \Gamma_f^{\text{out}},\\
\boldsymbol{u}_f(\boldsymbol{\mu}) &= \mathbf{0} && \text{on } \partial\Omega_f\setminus(\Gamma_f^{\text{in}}\cup\Gamma_f^{\text{out}}\cup\Gamma_S),\\
\boldsymbol{u}_f(\boldsymbol{\mu}) &= \boldsymbol{\lambda}_S(\boldsymbol{\mu}) && \text{on } \Gamma_S\,.
\end{aligned}
$$

### Darcy equations:

$$
\begin{aligned}
\nu(\boldsymbol{\mu})\mathbf{K}(\boldsymbol{\mu})^{-1}\boldsymbol{u}_p(\boldsymbol{\mu})+\nabla p_p(\boldsymbol{\mu}) &= \mathbf{0} && \text{in } \Omega_p,\\
\nabla\cdot\boldsymbol{u}_p(\boldsymbol{\mu}) &= 0 && \text{in } \Omega_p,\\
p_p(\boldsymbol{\mu}) &= 0 && \text{on } \Gamma_p^D,\\
\boldsymbol{u}_p(\boldsymbol{\mu})\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega_p\setminus(\Gamma_p^D\cup\Gamma_D),\\
p_p(\boldsymbol{\mu}) &= \lambda_D(\boldsymbol{\mu}) && \text{on } \Gamma_D.
\end{aligned}
$$

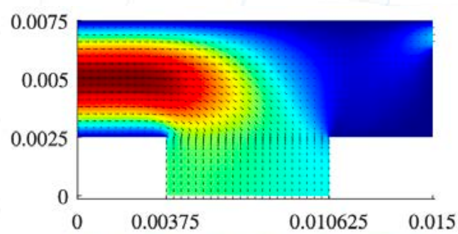
[Discacciati, Evans, Giacomini. arXiv:2507.06861]

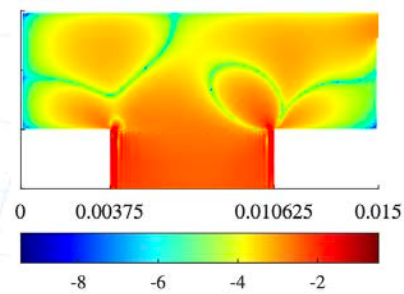


# Parametric Stokes-Darcy cross-flow

Max inlet velocity

Very permeable material

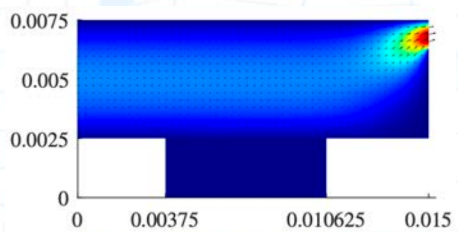


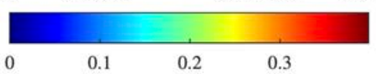



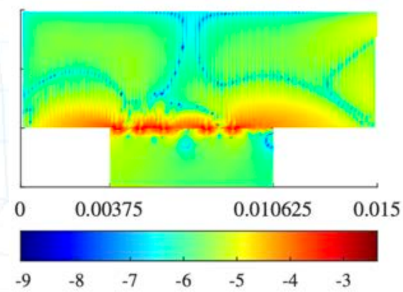

$\|\boldsymbol{u}^{\mathrm{PGD}}(\bar{\boldsymbol{\mu}})\|$ $\quad$ $\log_{10}(|\boldsymbol{u}^{\mathrm{PGD}}(\bar{\boldsymbol{\mu}})-\boldsymbol{u}^h(\bar{\boldsymbol{\mu}})|/\max_\Omega|\boldsymbol{u}^h(\bar{\boldsymbol{\mu}})|)$

Max inlet velocity

Almost impermeable material



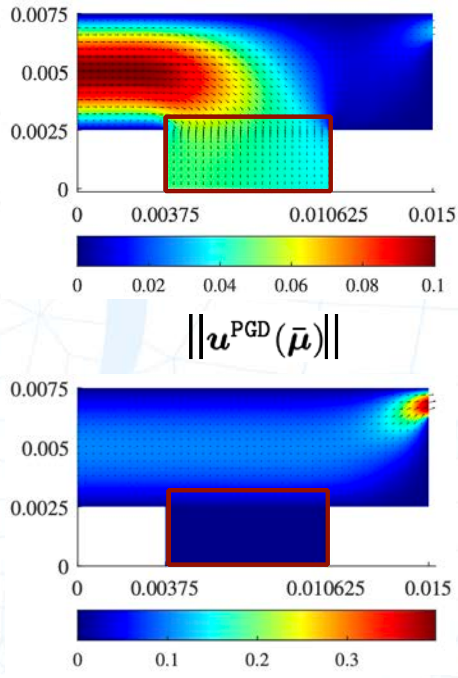




[Discacciati, Evans, Giacomini. arXiv:2507.06861]

## Parametric Stokes-Darcy cross-flow

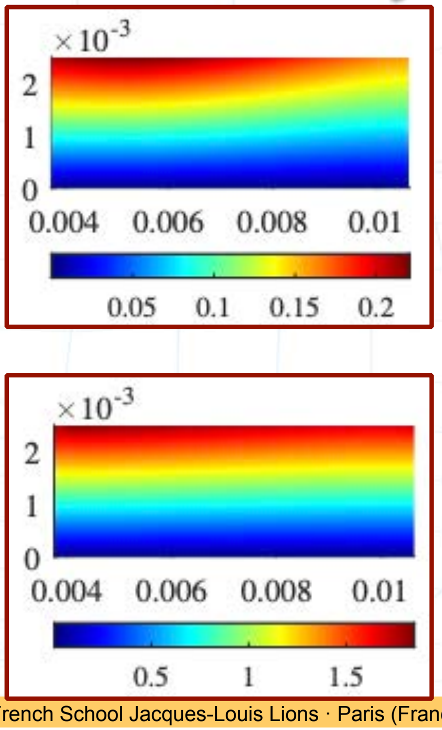
Max inlet velocity

Very permeable material




Max inlet velocity

Almost impermeable material




Offline phase: Local PGD (2 spatial and 1 parametric dimension)

Stokes
176 modes
67 minutes

119 minutes

Darcy
178 modes
52 minutes

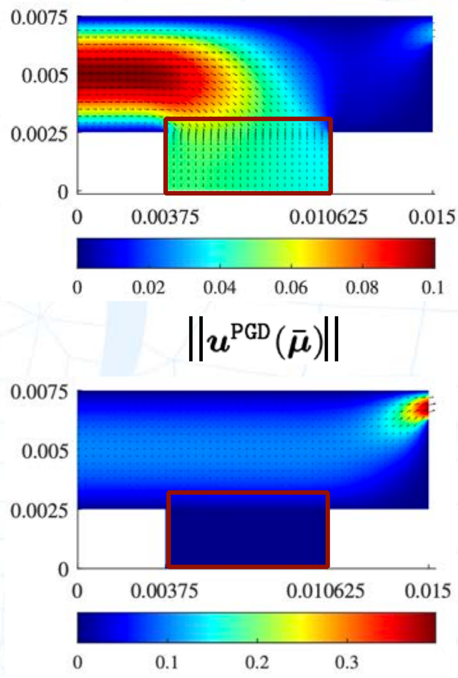
[Discacciati, Evans, Giacomini. arXiv:2507.06861]



## Parametric Stokes-Darcy cross-flow

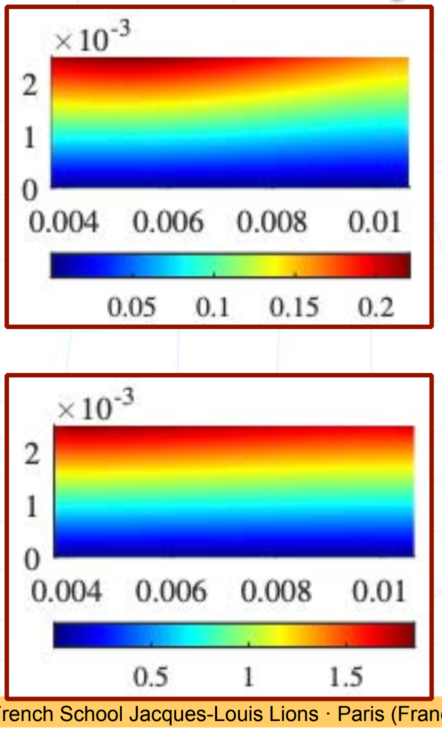
Max inlet velocity

Very permeable material




Max inlet velocity

Almost impermeable material




Online phase:
DD-PGD
144 GMRES iter.
**0.8 seconds**

DD-FEM
78 GMRES iter.
**24 minutes**

5 DD-FEM instances are as expensive as the offline phase!

[Discacciati, Evans, Giacomini. arXiv:2507.06861]

# Concluding remarks

## Real-time DD for parametric problems:

- Traces of FEM functions are used to define parametric Dirichlet BC in each subproblem (no auxiliary basis functions).
- Physics accounted for by the local PGD surrogate in each subdomain (also for multiple physics).
- ePGD provides a non-intrusive framework for physics-based surrogate models.
- No need for Lagrange multipliers, nor additional problems to be solved online to glue the subdomains.

Poisson & convection-diffusion - [Discacciati, Evans, Giacomini. CMAME, 2024]
Single VS. clustered nodes - [Discacciati, Evans, Giacomini. Finite Elem Anal Des, 2026]
Stokes & Stokes-Darcy - [Discacciati, Evans, Giacomini. arXiv:2507.06861]