Understanding Linelet Preconditioner through Spectral Analysis and High-Order Considerations

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Challenges with Highly Anisotropic Problems

Partial differential equations discretised on highly anisotropic meshes typically lead to linear systems with very large condition numbers. This severe ill-conditioning makes such systems challenging to solve: direct solvers become numerically unstable, while iterative methods require many iterations to converge.

At the same time, the increasing use of higher-order discretisations in practical applications further exacerbates these difficulties. These developments motivate a careful reassessment of preconditioning strategies that can maintain robustness and efficiency in strongly anisotropic and high-order settings.

i) Revisit the classical Linelet preconditioner through spectral analysis.

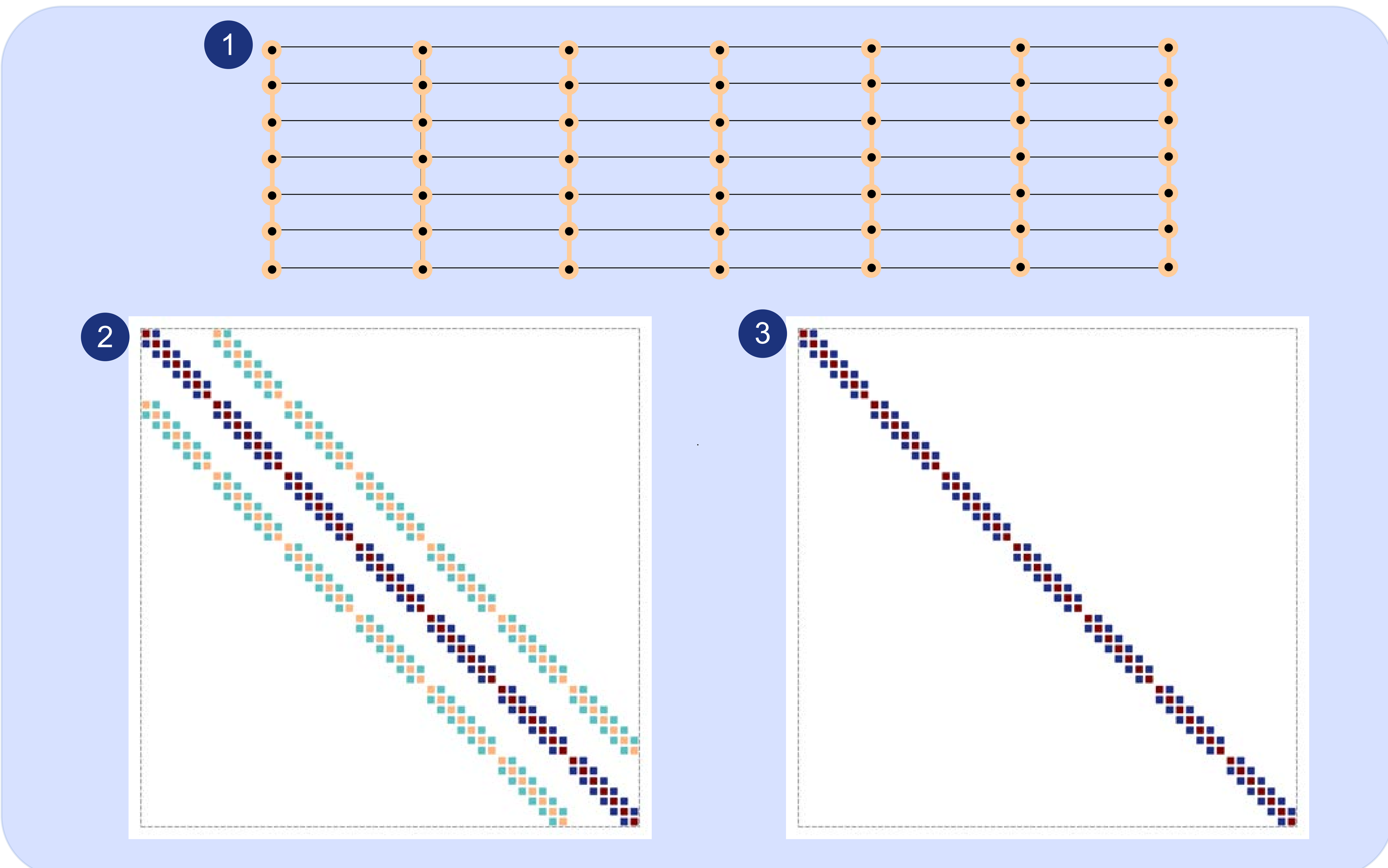
ii) Study the extension of Linelet preconditioner to higher-order problems.

Introduction to the Linelet Preconditioner

A classical approach for addressing boundary-layer problems is the **linelet preconditioner** (Soto et al., 2003).

The method proceeds as follows:

1. Mesh nodes are grouped into **linelets** based on geometric proximity or algebraic similarity (Olazabal et al., 2024).
2. The system matrix is **reordered** so that unknowns belonging to the same linelet are contigous.
3. The **tri-diagonal** system that contains interactions between contigous nodes is factrized and used to precondition the system.



This approach is effective as it directly resolves the dominant directional couplings induced by anisotropy.

Spectral Analisys for the Laplacian Operator

For a homogeneous structured mesh with Dirichlet boundary conditions, the discrete Laplacian operator admits an analytical spectral characterization. In this setting, the eigenvalues of the linelet-preconditioned system can also be derived explicitly.

$$\text{Unpreconditioned: } \lambda_{ij} = \frac{4}{3}\frac{h_y}{h_x}\sin^2\left(\frac{i\pi}{2N_x}\right)\left(2+\cos\left(\frac{j\pi}{N_y}\right)\right) + \frac{4}{3}\frac{h_x}{h_y}\left(2+\cos\left(\frac{i\pi}{N_x}\right)\right)\sin^2\left(\frac{j\pi}{2N_y}\right)$$

$$\text{Linelet Preconditioned: } \lambda_{ij} = 1+\cos\left(\frac{j\pi}{N_y}\right)\cdot\frac{2\frac{h_y}{h_x}\sin^2\left(\frac{i\pi}{2N_x}\right) - \frac{h_x}{h_y}\left(2+\cos\left(\frac{j\pi}{N_y}\right)\right)}{4\frac{h_y}{h_x}\sin^2\left(\frac{i\pi}{2N_x}\right) + \frac{h_x}{h_y}\left(2+\cos\left(\frac{j\pi}{N_y}\right)\right)}$$

These results enable a closed-form evaluation of the condition number of the preconditioned operator. We focus on the extreme anisotropic limit, in which the mesh stretching ratio tends to infinity.

$$\text{Unpreconditioned: } K \xrightarrow[\frac{h_y}{h_x}\to\infty]{} \frac{\sin^2\left(\frac{(N_x-1)\pi}{2N_x}\right)\left(2+\cos\left(\frac{\pi}{N_y}\right)\right)}{\sin^2\left(\frac{\pi}{2N_x}\right)\left(2+\cos\left(\frac{(N_y-1)\pi}{N_y}\right)\right)} \approx \boxed{3\cot^2\left(\frac{\pi}{2N_x}\right)}$$

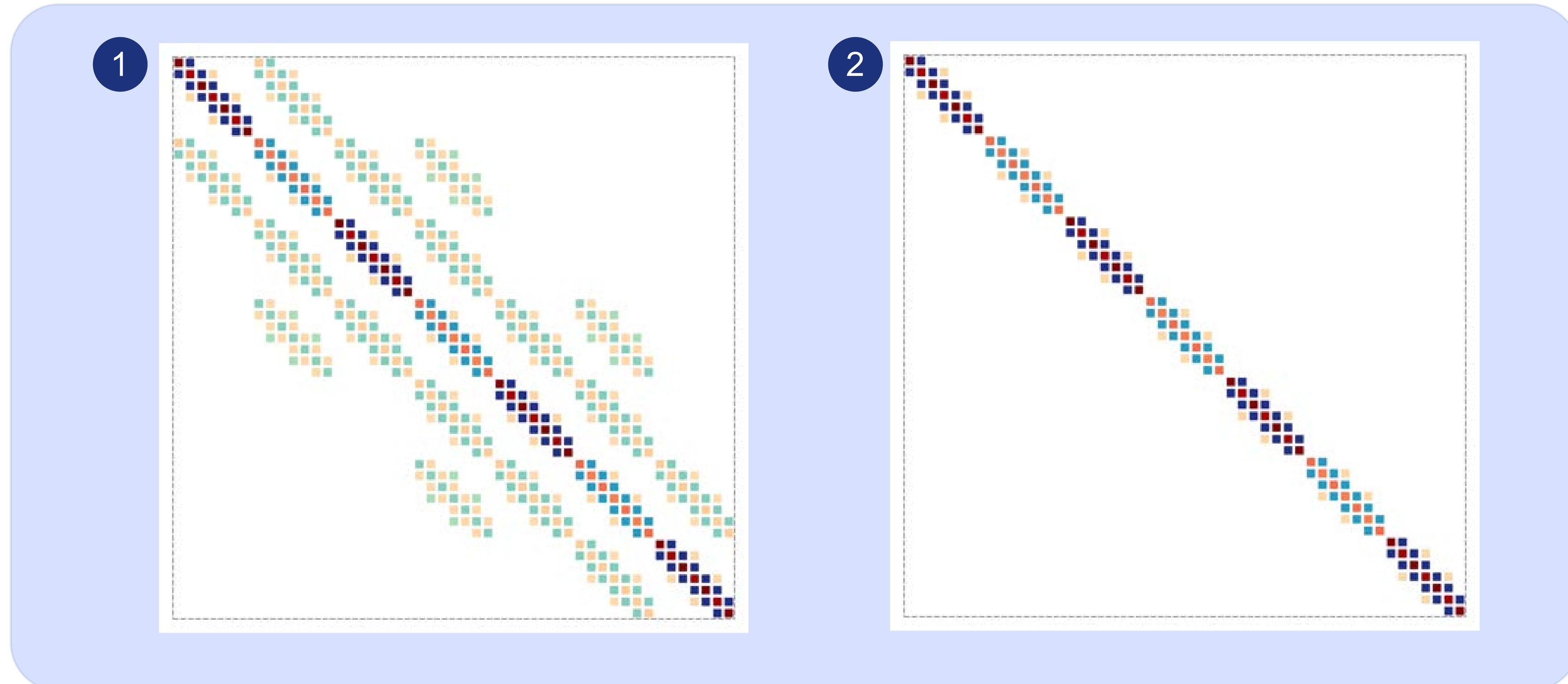
$$\text{Linelet Preconditioned: } K \xrightarrow[\frac{h_y}{h_x}\to\infty]{} \frac{2+\cos\left(\frac{\pi}{N_y}\right)}{2+\cos\left(\frac{(N_y-1)\pi}{N_y}\right)} \approx \boxed{3}$$

In this limit, the linelet preconditioner effectively eliminates the dependence of the condition number on the anisotropic direction.

Extension to Higher-Order Discretizations

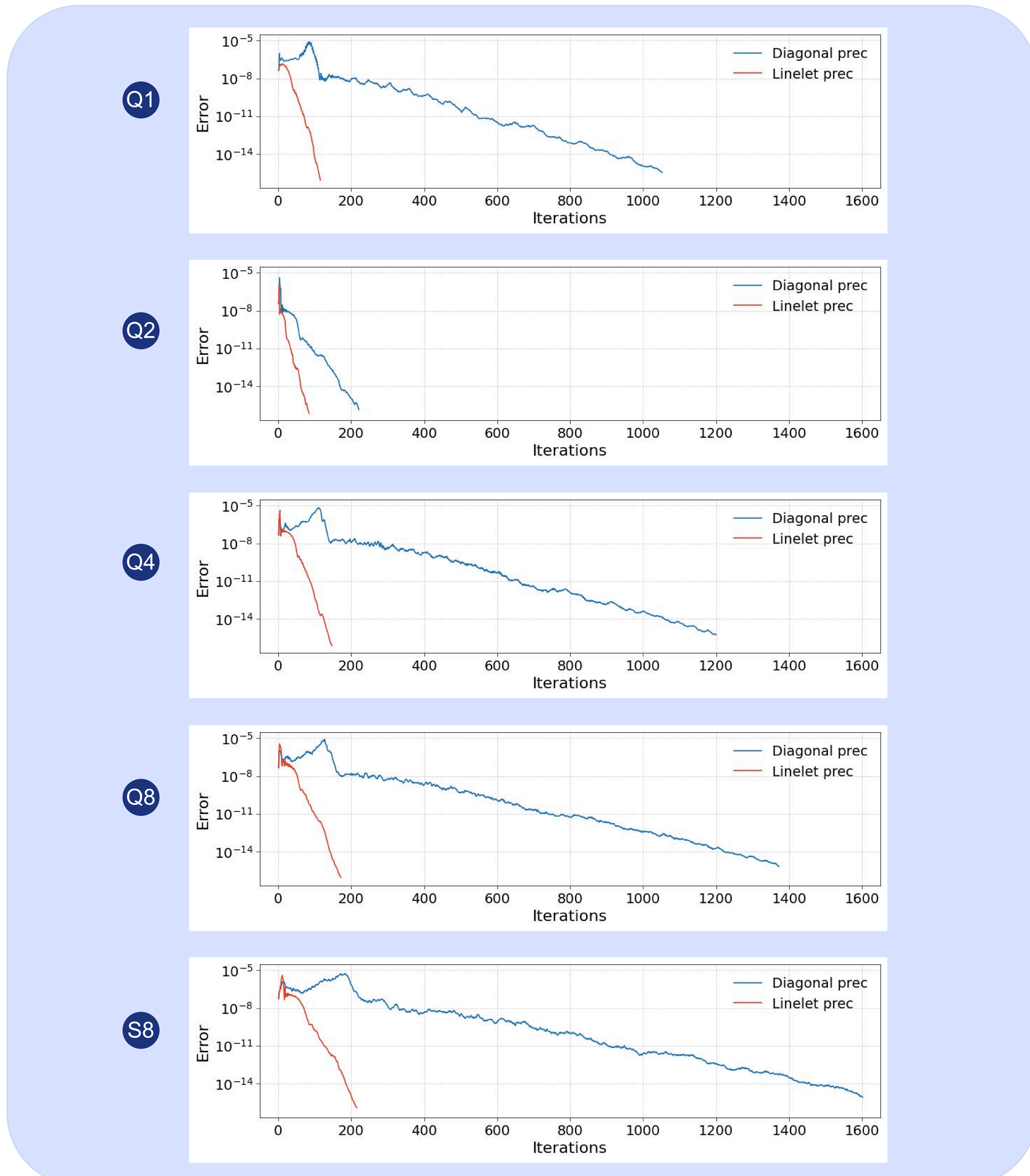
When the linelet preconditioner is applied to systems assembled using higher-order elements, the resulting preconditioning matrix may lose its symmetric positive definite (SPD) property, making it unsuitable for use with Conjugate Gradient solvers.

To overcome this issue, it is necessary to extend the factorization of the linelet-restricted system to include all higher-order intra-linelet couplings. This leads to banded block structures whose bandwidth increases with the polynomial degree: penta-diagonal blocks for second-order elements (Ahrabi & Mavriplis, 2018), hepta-diagonal blocks for third-order elements, and, in general, wider banded blocks for higher-order discretizations.



Numerical evaluation on realistic anisotropic test cases

To assess the effectiveness of the linelet preconditioner for higher-order discretizations, we study the error evolution of a Laplacian problem on a non-uniformly stretched mesh. Results are presented for different polynomial orders while keeping the total number of degrees of freedom constant, enabling a fair comparison across discretization orders.



Conclusions

The linelet preconditioner provides an effective strategy for mitigating the ill-conditioning induced by highly anisotropic meshes. For linear discretizations, its spectral properties explain the observed improvement in solver convergence. Although a direct analytical characterization is no longer feasible for higher-order elements, numerical results show that preserving all intra-linelet couplings is essential to maintain stability and effectiveness. When properly extended, the linelet preconditioner retains its ability to control conditioning and delivers performance comparable to the linear case, even in strongly anisotropic and high-order settings.

References

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