FETI Methods

FETI-1LM

François-Xavier Roux

ONERA and LJLL (retired)

FETI: main features

First method based on Lagrange multipliers

$\Rightarrow$ discontinuous solution (non conforming method)

$\Rightarrow$ dual interface variables
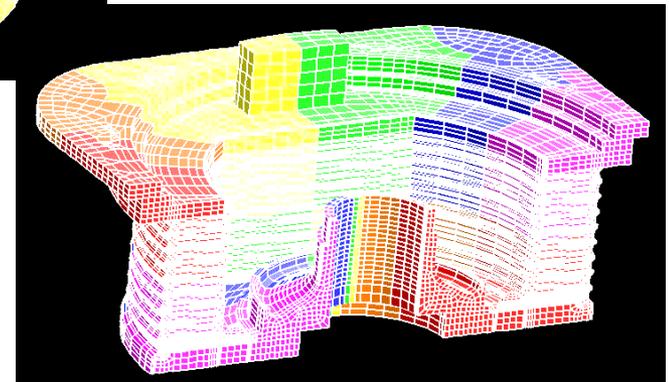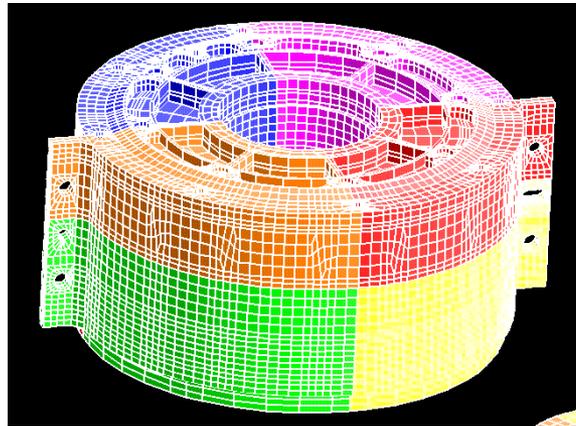
Easy interface management

Imbedded coarse grid preconditioner

Early motivation

Solution of large ill-conditioned problem in structural analysis

Parallelization in message passing programming environment

Complex geometry
Composite materials

PDE, variational formulations

Partial differential equations

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ + \text{ boundary conditions on } \partial\Omega \end{cases}$$

Stokes formula

$$\int_\Omega \nabla u \,.\, \nabla v = \int_\Omega fv + \int_{\partial\Omega} \frac{\partial u}{\partial n} v$$

Variational formulation of Dirichlet problem

$$u = u_D \text{ on } \partial\Omega \quad \int_\Omega \nabla u \,.\, \nabla v = \int_\Omega fv \quad \forall v / v = 0 \text{ on } \partial\Omega$$

Variational formulation of Neumann problem

$$\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega \quad \int_\Omega \nabla u \,.\, \nabla v = \int_\Omega fv + \int_{\partial\Omega} gv \quad \forall v$$

FE discretization, Dirichlet and Neumann problems

Block structure with inner and boundary unknowns

$$\begin{pmatrix} K_{ii} & K_{ib} \\ K_{bi} & K_{bb} \end{pmatrix} \ , \ \begin{pmatrix} x_i \\ x_b \end{pmatrix} \ , \ \begin{pmatrix} b_i \\ b_b \end{pmatrix}$$

Dirichlet problem

$$\begin{cases} K_{ii}x_i + K_{ib}x_b = b_i \\ \qquad\quad x_b = x_b \end{cases}$$

Neumann problem

$$\begin{pmatrix} K_{ii} & K_{ib} \\ K_{bi} & K_{bb} \end{pmatrix}\begin{pmatrix} x_i \\ x_b \end{pmatrix} = \begin{pmatrix} b_i \\ b_b + \lambda_b \end{pmatrix}$$

Solution of Dirichlet problem

$$\begin{cases} K_{ii}x_i + K_{ib}x_b = b_i \\ \qquad x_b = x_b \end{cases} \quad \Rightarrow \quad K_{ii}x_i = b_i - K_{ib}x_b \quad \Rightarrow \quad x_i = -K_{ii}^{-1}K_{ib}x_b + K_{ii}^{-1}b_i$$

Neumann problem with the same solution

$$K_{bi}x_i + K_{bb}x_b = K_{bb}x_b - K_{bi}K_{ii}^{-1}K_{ib}x_b + K_{bi}K_{ii}^{-1}b_i$$

$$\Rightarrow \begin{pmatrix} K_{ii} & K_{ib} \\ K_{bi} & K_{bb} \end{pmatrix} \begin{pmatrix} x_i \\ x_b \end{pmatrix} = \begin{pmatrix} b_i \\ (K_{bb} - K_{bi}K_{ii}^{-1}K_{ib})x_b + K_{bi}K_{ii}^{-1}b_i \end{pmatrix} = \begin{pmatrix} b_i \\ b_b + \lambda_b \end{pmatrix}$$

$$\boxed{\begin{aligned} \lambda_b &= \left(K_{bb} - K_{bi}K_{ii}^{-1}K_{ib}\right)x_b - \left(b_b - K_{bi}K_{ii}^{-1}b_i\right) \\ S &= \left(K_{bb} - K_{bi}K_{ii}^{-1}K_{ib}\right) \\ \lambda_b &= Sx_b - c_b \end{aligned}}$$

The Schur complement is the mapping of the trace on the flux on the boundary for a field satisfying the equations inside the domain
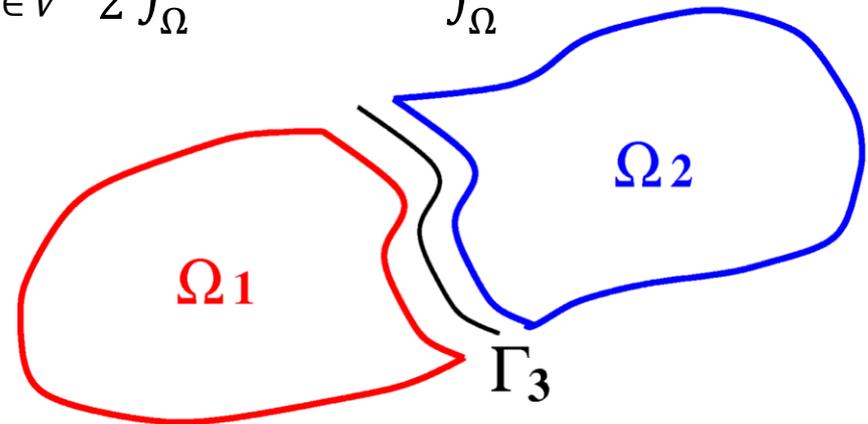
Energy formulation of linear elasticity

Linear elasticity problem

$$\begin{cases} div\ \sigma + f = 0 \ \ \textbf{in}\ \Omega \\ \sigma = A(\varepsilon) \ \ \textbf{in}\ \Omega \\ +\ \text{boundary conditions on}\ \partial\Omega \end{cases}$$

Minimization problem

$$u = Arg\ \underset{v\,\in\,V}{Min}\ \frac{1}{2}\int_{\Omega} \sigma(v)\,\nabla v - \int_{\Omega} f\,v$$

Minimization problem under constraint for the 2-domain problem

$$(u_1, u_2) = Arg\ \underset{\substack{(v_1,v_2)\in V_1 X V_2 \\ v_1 = v_2\ \text{on}\ \Gamma_3}}{Min}\ \underbrace{\frac{1}{2}\int_{\Omega_1} \sigma_1(v_1)\,\nabla v_1 - \int_{\Omega_1} f_1\,v_1}_{J_1(u_1)} + \underbrace{\frac{1}{2}\int_{\Omega_2} \sigma_2(v_2)\,\nabla v_2 - \int_{\Omega_2} f_2\,v_2}_{J_2(u_2)}$$

$J_1(u_1)$ + $J_2(u_2)$

Ω_2

Ω_1

Γ_3

Mixed formulation

Saddle point of Lagrangian

$$\mathcal{L}(v_1, v_2, \mu) = J_1(v_1) + J_2(v_2) + \int_{\Gamma_3} (v_1 - v_2)\mu$$

$$(u_1, u_2, \lambda) = Arg \min_{(v_1,v_2) \in V_1 X V_2} \max_{\mu \in W} \mathrm{L}(v_1, v_2, \mu)$$

Variational formulation of saddle point problem

$$\int_{\Omega_1} \sigma_1(u_1) \nabla v_1 = \int_{\Omega_1} f_1 v_1 - \int_{\Gamma_3} \lambda v_1 \quad \forall v_1 \in V_1$$

$$\int_{\Omega_2} \sigma_2(u_2) \nabla v_2 = \int_{\Omega_2} f_2 v_2 + \int_{\Gamma_3} \lambda v_2 \quad \forall v_2 \in V_2$$

$$\int_{\Gamma_3} (u_1 - u_2)\mu = 0 \quad \forall \mu \in W$$
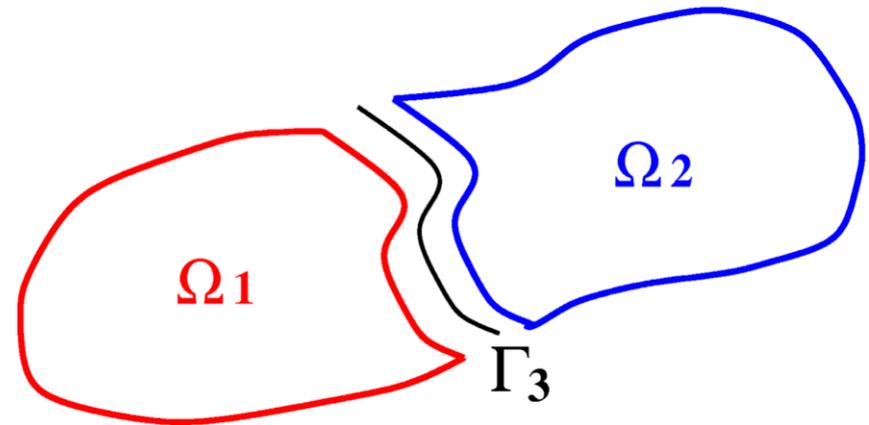
Interpretation of mixed formulation

Local problems

$$\begin{cases} div\sigma_1 = f_1 \ \text{ in } \Omega_1 \\ + \text{ boundary conditions on } \partial\Omega_1 \cap \partial\Omega \\ \sigma_1 n_1 = -\lambda \text{ on } \Gamma_3 \end{cases}$$

$$\begin{cases} div \ \sigma_2 = f_2 \ \text{ in } \Omega_2 \\ + \text{ boundary conditions on } \partial\Omega_2 \cap \ \partial\Omega \\ \sigma_2 n_2 = \lambda \text{ on } \Gamma_3 \end{cases}$$

Interface conditions

$$\begin{cases} u_1 = u_2 \text{ on } \Gamma_3 \\ \sigma_1 n_1 + \sigma_2 n_2 = 0 \ \text{ on } \Gamma_3 \end{cases}$$

FETI method :

$$\begin{cases} -\sigma_1 n_1 = \sigma_2 n_2 = \lambda \\ F\lambda - d = u_1 - u_2 = 0 \end{cases}$$

Global homogeneous Dirichlet problem

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v \quad \forall v / v = 0 \text{ on } \partial\Omega$$
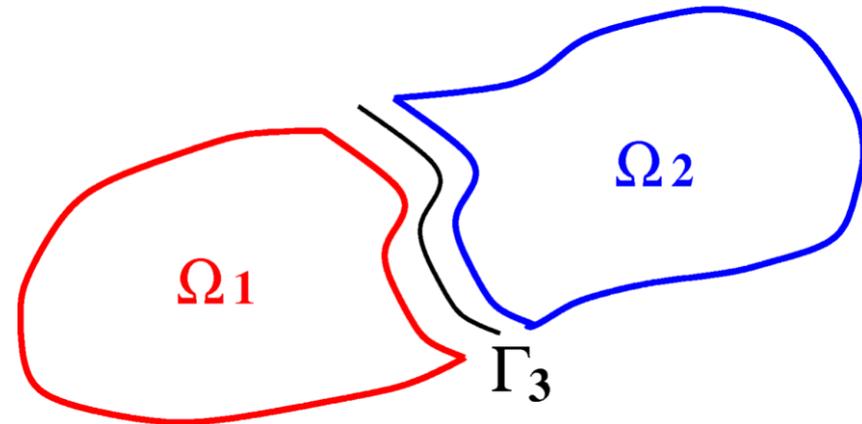
Local Stokes formula

$$\int_{\Omega_1} \nabla u_1 \cdot \nabla v_1 = \int_{\Omega_1} f v_1 + \int_{\Gamma_3} \frac{\partial u_1}{\partial n_1} v_1 \quad \forall v_1 / v_1 = 0 \text{ on } \partial\Omega_1 \setminus \Gamma_3$$

$$\int_{\Omega_2} \nabla u_2 \cdot \nabla v_2 = \int_{\Omega_2} f v_2 + \int_{\Gamma_3} \frac{\partial u_2}{\partial n_2} v_2 \quad \forall v_2 / v_2 = 0 \text{ on } \partial\Omega_2 \setminus \Gamma_3$$

Interface conditions:
continuity of field and balance of flux

$$\begin{cases} u_1 = u_2 \text{ on } \Gamma_3 \\ \dfrac{\partial u_1}{\partial n_1} + \dfrac{\partial u_2}{\partial n_2} = 0 \text{ on } \Gamma_3 \end{cases}$$
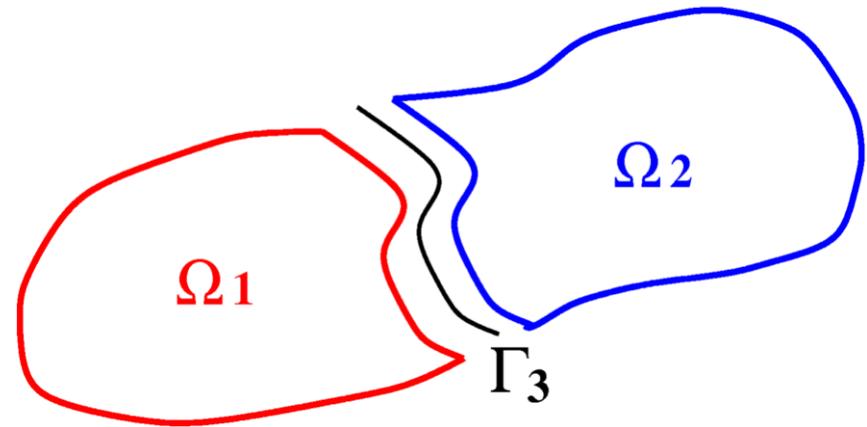
Global partial differential equations

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega \\ + \text{boundary conditions on } \partial\Omega \end{cases}$$

Local partial differential equations

$$\begin{cases} -\Delta u_1 = f_1 \quad \text{in } \Omega_1 \\ + \text{boundary conditions on } \partial\Omega \cap \partial\Omega_1 \end{cases}$$

$$\begin{cases} -\Delta u_2 = f_2 \quad \text{in } \Omega_2 \\ + \text{boundary conditions on } \partial\Omega \cap \partial\Omega_2 \end{cases}$$
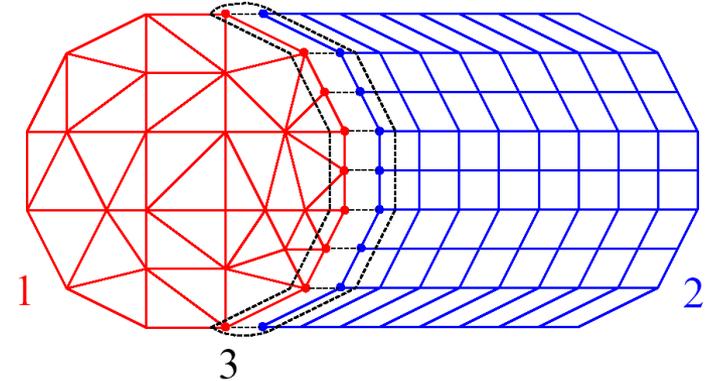

Interface conditions: continuity of field and balance of flux

$$\begin{cases} u_1 = u_2 \text{ on } \Gamma_3 \\ \dfrac{\partial u_1}{\partial n_1} + \dfrac{\partial u_2}{\partial n_2} = 0 \quad \text{on } \Gamma_3 \end{cases}$$

Domain decomposition methods, algebraic approach

Global system of equations

$$\begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Local systems of equations

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33}^{(1)} \end{pmatrix} \ , \ \begin{pmatrix} x_1 \\ x_3^{(1)} \end{pmatrix} \ , \ \begin{pmatrix} b_1 \\ b_3^{(1)} \end{pmatrix} \qquad \begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33}^{(2)} \end{pmatrix} \ , \ \begin{pmatrix} x_2 \\ x_3^{(2)} \end{pmatrix} \ , \ \begin{pmatrix} b_2 \\ b_3^{(2)} \end{pmatrix}$$
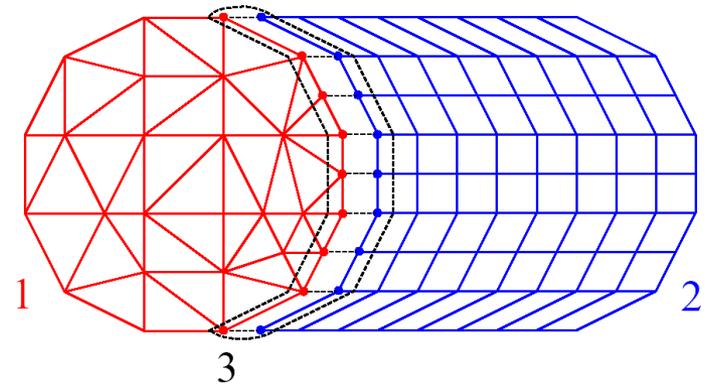
$$K_{33}^{(1)} + K_{33}^{(2)} = K_{33}$$

$$b_3^{(1)} + b_3^{(2)} = b_3$$

Local inner equations

$$K_{11}\, x_1 + K_{13}\, x_3^{(1)} = b_1$$

$$K_{22}\, x_2 + K_{23}\, x_3^{(2)} = b_2$$

Admissibility condition at interface

$$x_3^{(1)} = x_3^{(2)} \quad (= x_3)$$

Equilibrium condition at interface

$$K_{31}\, x_1 + K_{32}\, x_2 + K_{33}\, x_3 = b_3$$

$$K_{31}\, x_1 + K_{33}^{(1)}\, x_3^{(1)} - b_3^{(1)} + K_{32}\, x_2 + K_{33}^{(2)}\, x_3^{(2)} - b_3^{(2)} = 0$$

Local Dirichlet to Neumann mapping

Solution of inner local problems whatever the boundary conditions on interface

$$K_{11}\, x_1 = b_1 - K_{13}\, x_3^{(1)} \qquad\qquad K_{22}\, x_2 = b_2 - K_{23}\, x_3^{(2)}$$

Local contributions to residual : non zero only on interface

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3^{(1)} \end{pmatrix} - \begin{pmatrix} b_1 \\ b_3^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \left(K_{33}^{(1)} - K_{31} K_{11}^{-1} K_{13} \right) x_3^{(1)} - \left(b_3^{(1)} - K_{31} K_{11}^{-1} b_1 \right) \end{pmatrix}$$

$$\begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33}^{(2)} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3^{(2)} \end{pmatrix} - \begin{pmatrix} b_2 \\ b_3^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ \left(K_{33}^{(2)} - K_{32} K_{22}^{-1} K_{23} \right) x_3^{(2)} - \left(b_3^{(2)} - K_{32} K_{22}^{-1} b_2 \right) \end{pmatrix}$$

Local problems with Neumann boundary conditions

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3^{(1)} + \lambda_1 \end{pmatrix} \qquad\qquad \begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33}^{(2)} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3^{(2)} \end{pmatrix} = \begin{pmatrix} b_2 \\ b_3^{(2)} + \lambda_2 \end{pmatrix}$$

Condensed interface problems

Admissibility condition at interface

$$x_3^{(1)} - x_3^{(2)} = 0$$

Equilibrium condition at interface

$$K_{31}\, x_1 + K_{33}^{(1)}\, x_3^{(1)} - b_3^{(1)} + K_{32}\, x_2 + K_{33}^{(2)}\, x_3^{(2)} - b_3^{(2)} = 0$$

$$\lambda_1 + \lambda_2 = 0$$

Solution of local Dirichlet or Neumann problems

$$\lambda_1 = S_1 x_3^{(1)} - c_3^{(1)} \qquad\qquad \lambda_2 = S_2 x_3^{(2)} - c_3^{(2)}$$
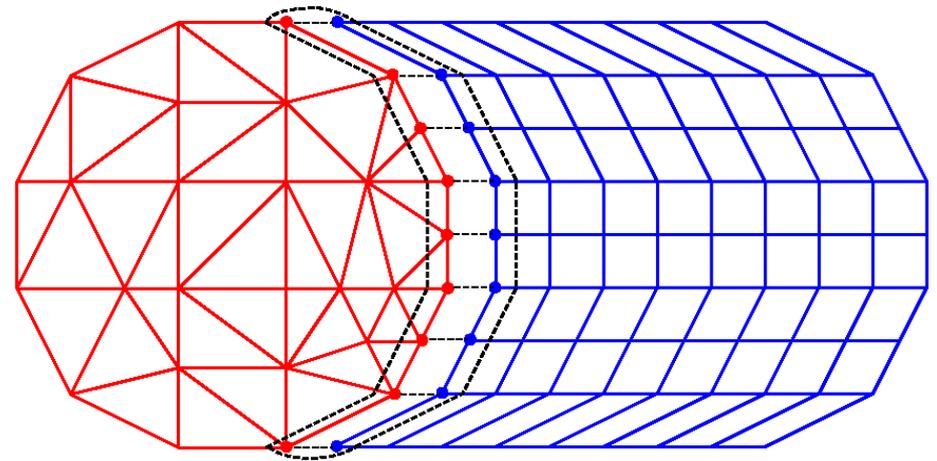
Schur complement method

Interface unknown: $\quad x_3 = x_3^{(1)} = x_3^{(2)}$

Solution of local Dirichlet problems in subdomains

$$K_{11}\, x_1 = b_1 - K_{13}\, x_3$$

$$K_{22}\, x_2 = b_2 - K_{23}\, x_3$$

Equilibrium condition on interface => definition of residual

$$\lambda_1 + \lambda_2 = S_1 x_3 - c_3^{(1)} + S_2 x_3 - c_3^{(2)} = S x_3 - c_3 = 0$$

Implementation

Solution of local system via a direct method (Gauss, Cholesky)

$$K_{11}\, x_1 = b_1 - K_{13}\, x_3 \qquad K_{22}\, x_2 = b_2 - K_{23}\, x_3$$

Local contribution to interface residual

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_3^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ S_1 x_3 - c_3^{(1)} \end{pmatrix} \qquad \begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33}^{(2)} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} b_2 \\ b_3^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ S_2 x_3 - c_3^{(2)} \end{pmatrix}$$

Global interface residual obtained by assembling local contributions

$$S_1 x_3 - c_3^{(1)} + S_2 x_3 - c_3^{(2)} = (S_1 + S_2) x_3 - (c_3^{(1)} + c_3^{(2)}) = S x_3 - c_3$$

Conjugate gradient iterations for the condensed interface problem

Very well suited method for distributed memory parallel system with message passing programming environment

FETI method 2 subdomains

Interface unknown: $\qquad \lambda = \lambda_1 = -\lambda_2$

Solution of local Neumann problems in subdomains

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3^{(1)} + \lambda_1 \end{pmatrix} \qquad \begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33}^{(2)} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3^{(2)} \end{pmatrix} = \begin{pmatrix} b_2 \\ b_3^{(2)} + \lambda_2 \end{pmatrix}$$

Interface residual :

$$
\begin{aligned}
x_3^{(1)} - x_3^{(2)} \quad &= S_1^{-1}(\lambda_1 - c_3^{(1)}) - S_2^{-1}(\lambda_2 - c_3^{(2)}) \\
&= (S_1^{-1} + S_2^{-1})\lambda - (S_1^{-1} c_3^{(1)} - S_2^{-1} c_3^{(2)}) \\
&= F\lambda - d
\end{aligned}
$$

S and F have similar condition numbers but small eigenvalues of S correspond to large eigenvalues of F: Krylov methods converge faster for F

FETI method many subdomains

Local trace operator on interfaces of subdomain $\Omega_s : B_s$
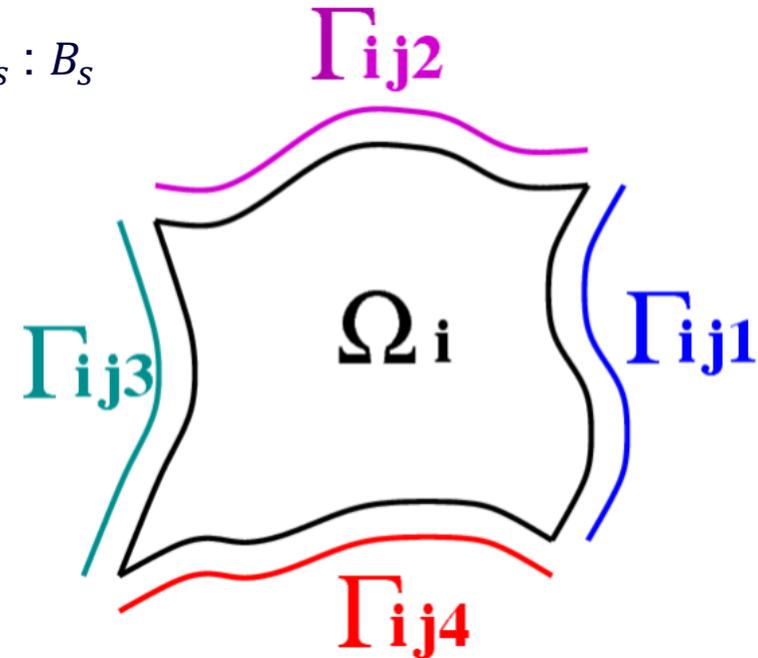
Global jump of trace operator on interfaces :

$$\sum_{s=1}^{n_s} B_s u_s = u_i - u_j \;\; \text{on} \;\; \Gamma_{ij} = \Omega_i \cap \Omega_j$$

Local problem in subdomain Ω_s

$$K_s u_s = b_s + B_s^t \lambda \;\Rightarrow\; u_s = K_s^{-1}\big(b_s + B_s^t \lambda\big)$$

Condensed FETI problem on interfaces

$$\sum_{s=1}^{n_s} B_s u_s = 0 \Rightarrow \sum_{s=1}^{n_s} B_s K_s^{-1} B_s^t \lambda = - \sum_{s=1}^{n_s} B_s u_s^0 \quad \text{with} \;\; K_s u_s^0 = b_s$$

FETI method many subdomains

Local trace operator on interfaces of subdomain $\Omega_s : B_s = T_s R_s$

Global jump of trace operator on interfaces :

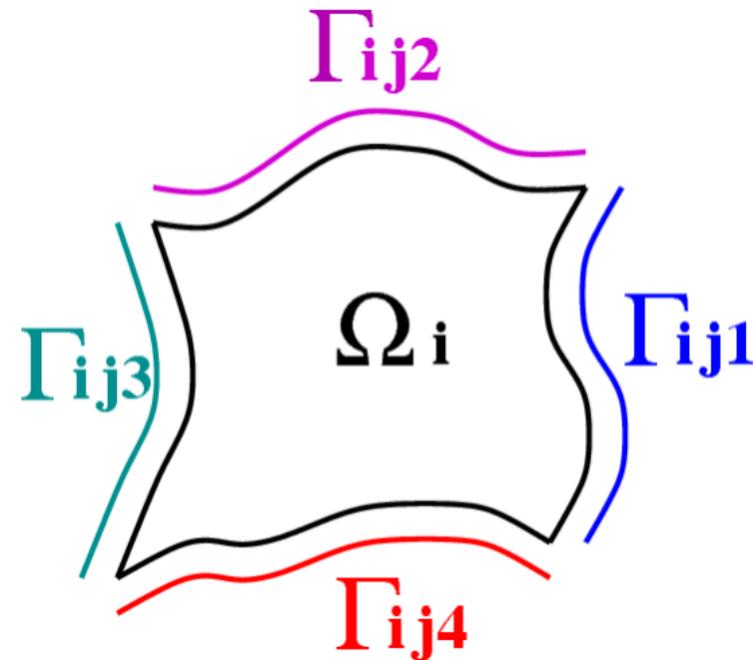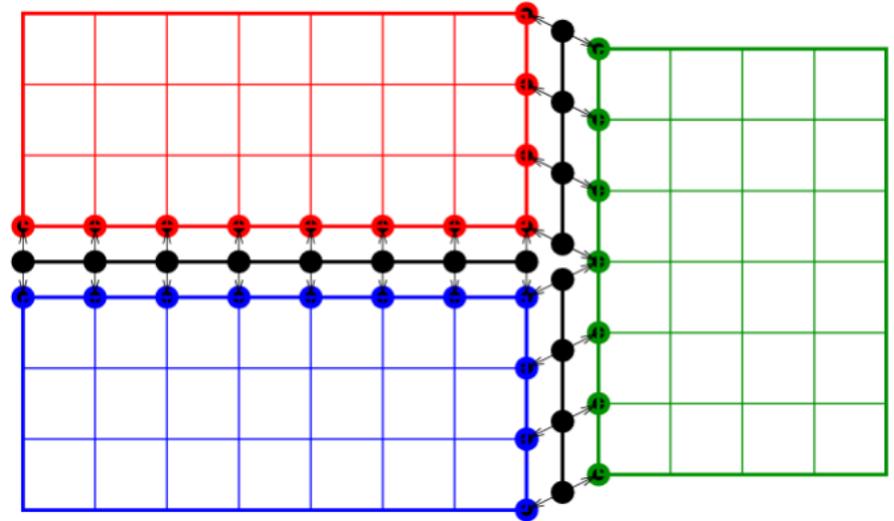$$\sum_{s=1}^{n_s} B_s u_s = R_i u_i - R_j u_j \ \text{ on } \Gamma_{ij} = \Omega_i \cap \Omega_j$$

Local problem in subdomain Ω_s

$$K_s u_s = b_s + B_s^t \lambda \ \Rightarrow \ u_s = K_s^{-1}\big(b_s + B_s^t \lambda\big)$$

$$S_s R_s u_s = c_s + T_s^t \lambda \ \Rightarrow \ R_s u_s = S_s^{-1}\big(c_s + T_s^t \lambda\big)$$
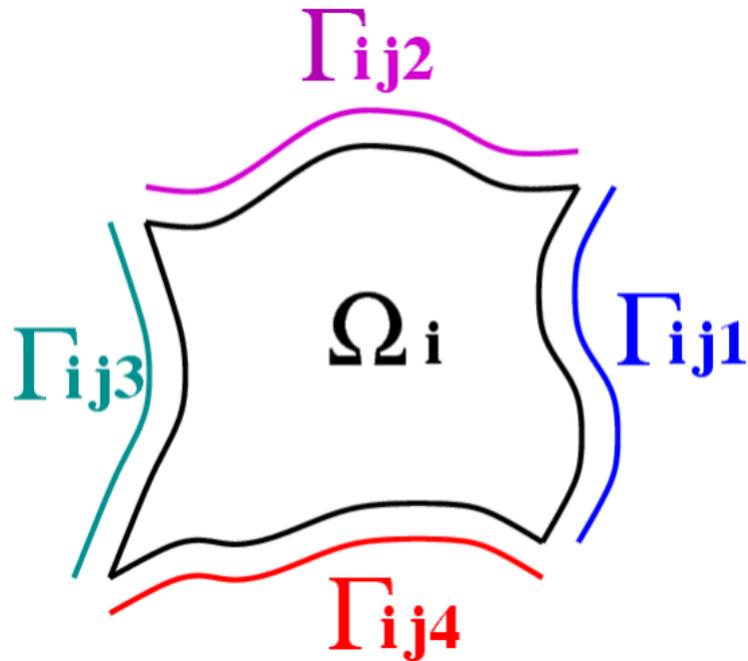
Condensed FETI problem on interfaces

$$\sum_{s=1}^{n_s} B_s u_s = 0 \Rightarrow \sum_{s=1}^{n_s} T_s S_s^{-1} T_s^t \lambda = - \sum_{s=1}^{n_s} T_s R_s u_s^0 \quad \text{with} \quad S_s R_s u_s^0 = c_s$$

Easy interface management

No multiple interface unknowns

Uncoupled computation of jump of interface displacements

Very simple implementation in message passing programming environment

Same implementation for 2 or n_s subdomains

Solution of singular mixed problem

Solution via condensation (Uzawa)

$$\begin{pmatrix} K & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \implies B\,K^{-1}B^t\,\lambda = B\,K^{-1}\,b$$

$$K = \begin{pmatrix} K_1 & & & \\ & K_2 & & \\ & & \ddots & \\ & & & K_{n_s} \end{pmatrix}$$

$$B\,K^{-1}B^t\lambda = 0 \Rightarrow (B\,K^{-1}B^t\lambda, \lambda) = (K^{-1}B^t\lambda, B^t\lambda) = 0$$

If *K* is SPD, *Ker(BK⁻¹Bᵗ) = Ker(Bᵗ)* and so *Im(BK⁻¹Bᵗ) = Im(B)*

CG converges to unique solution that belongs to *Im(BK⁻¹Bᵗ) = Im(B)*

No problem with redundant continuity conditions with FETI

FETI solution is identical to global solution : LBB condition not required

About the orientation of interfaces

Very easy implementation in message passing programming environment
Each subdomain uses its outer normal orientation
The jump of displacements is defined locally

$$[u] = u_{local} - u_{external}$$

Local Lagrange multipliers defined on both sides automatically satisfy

$$\lambda_{local} = -\lambda_{external}$$

Same thing for all interface vectors built in the projected conjugate gradient method

Dot products are the same (minus x minus = plus)

Optimal local preconditioner for FETI method

FETI condensed interface operator

$$F = \left(S_1^{-1} + S_2^{-1} \right)$$

Optimal local preconditioner (convergence independent of mesh size)

$$\tilde{F}^{-1} = \left(d_1 S_1 d_1 + d_2 S_2 d_2 \right)$$

$$d_1 + d_2 = 1$$

Computation of preconditioned gradient

$$K_{11} v_1 = -K_{13} \tfrac{1}{2} g \qquad \begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} v_1 \\ \tfrac{1}{2} g \end{pmatrix} = \begin{pmatrix} 0 \\ S_1 \tfrac{1}{2} g \end{pmatrix}$$

$$K_{22} v_2 = -K_{23} \tfrac{1}{2} g \qquad \begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33}^{(2)} \end{pmatrix} \begin{pmatrix} v_2 \\ \tfrac{1}{2} g \end{pmatrix} = \begin{pmatrix} 0 \\ S_2 \tfrac{1}{2} g \end{pmatrix}$$

Solution of Dirichlet problem in each subdomain

Mechanical interpretation of weighting

The interface gradient is the jump of displacements
Find the interaction force which will make the gap to vanish

Impose each subdomain interface to move a fraction of the gap, compute the local force to impose
(Dirichlet to Neumann)

Take a weighted average of the
Responses from both subdomain on
each interface

Stiff subdomain moves less than smooth one…

Condition number independent of mesh size (h), depends on number of subdomains (H)
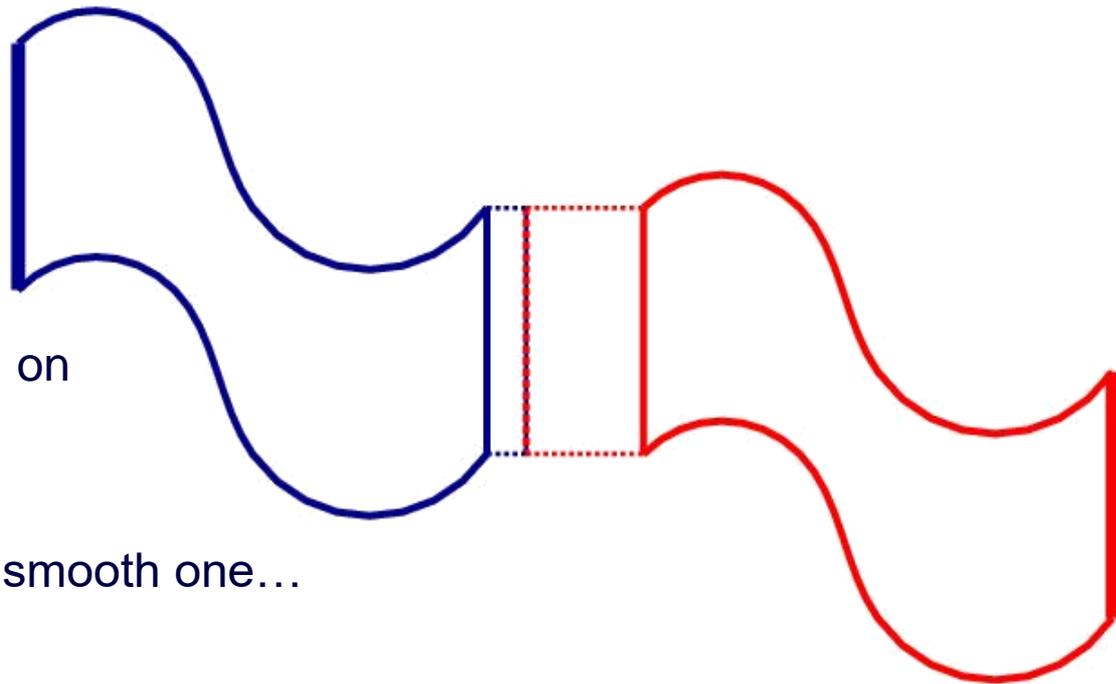
FETI projection

Ill posed local Neumann problem

How to solve the problem using a direct solver ?

FETI with ill posed local Neumann problems

Ill posed local Neumann problem

Null space (rigid body motions in elasticity, constant for Laplace)

Local problem in subdomain Ω_s

$$K_s u_s = b_s + B_s^t \lambda \;\Rightarrow\; \begin{cases} u_s = K_s^+\left(b_s + B_s^t \lambda\right) + N_s \alpha_s \\ \qquad N_s^t\left(b_s + B_s^t \lambda\right) = 0 \end{cases}$$

Condensed FETI problem on interfaces

$$\sum_{s=1}^{n_s} B_s u_s = 0 \Rightarrow \sum_{s=1}^{n_s} B_s K_s^+ B_s^t \lambda + \sum_{s=1}^{n_s} B_s N_s \alpha_s = -\sum_{s=1}^{n_s} B_s K_s^+ b_s$$

Mixed FETI problem

Mixed condensed interface problem

$$\begin{pmatrix} F & G \\ G^t & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \alpha \end{pmatrix} = \begin{pmatrix} d \\ c \end{pmatrix}$$

$$F = \sum_{s=1}^{n_s} B_s K_s^+ B_s^t \qquad d = -\sum_{s=1}^{n_s} B_s K_s^+ b_s$$

$$G = \sum_{s=1}^{n_s} B_s N_s \qquad c_s = -N_s^t b_s$$

Orthogonal projection in the kernel of G^t

$$P\mu = \mu + G\beta \qquad G^t P\mu = 0 \implies G^t G\beta = -G^t \mu$$

$$P \; projection \; in \; Ker(G^t)$$
$$P = I - G(G^t G)^{-1} G^t$$

FETI projection

Solution of the mixed problem via the projected CG ?

Computation of FETI projection

Block $\left(G^t G\right)_{ij} = (B_i N_i)^t (B_j N_j)$

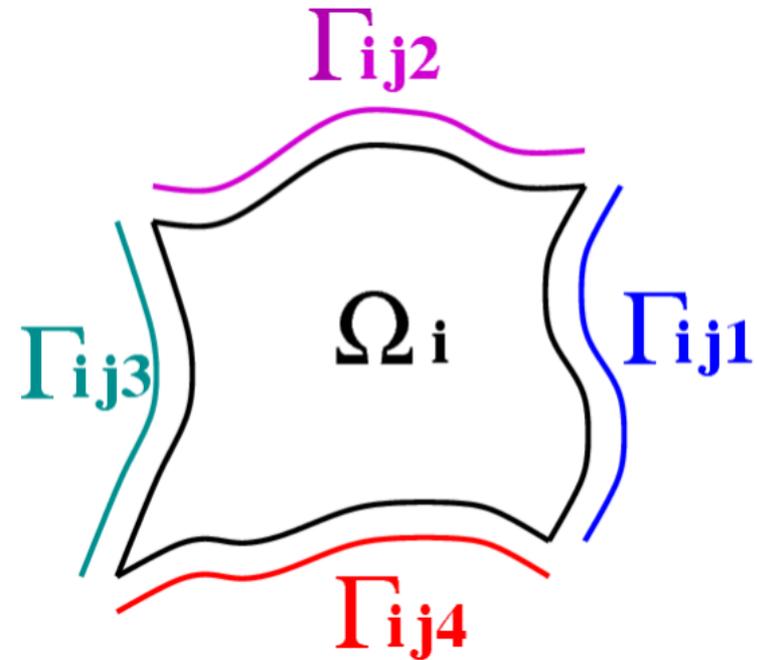Null if subdomains Ω_i and Ω_j are not neighbors

Contribution of interface Γ_{ij} to matrix $G^t G$

Contribution to diagonal block (i, i)
$$(N_i . N_i)_{\Gamma_{ij}}$$

Contribution to off-diagonal block (i, j)
$$-\left(N_i . N_j\right)_{\Gamma_{ij}}$$

Matrix $G^t G$ is sparse with the structure of a finite element matrix whose elements are the subdomains and whose degrees of freedom are the null spaces components

Small dimension, easy to factorize

Property of FETI projection

The FETI gradient is the jump of local solutions obtained with the pseudo-inverses on interfaces

$$g = \sum_{s=1}^{n_s} B_s K_s^+ \left(b_s + B_s^t \lambda \right)$$

The FETI projection of the gradient computes the null spaces components which minimize the jump

$$Pg = \sum_{s=1}^{n_s} B_s K_s^+ \left(b_s + B_s^t \lambda \right) + \sum_{s=1}^{n_s} B_s N_s \alpha_s$$

Global « coarse grid » preconditioner

Condition number almost independent of number of subdomains (logarithmic dependence on H)
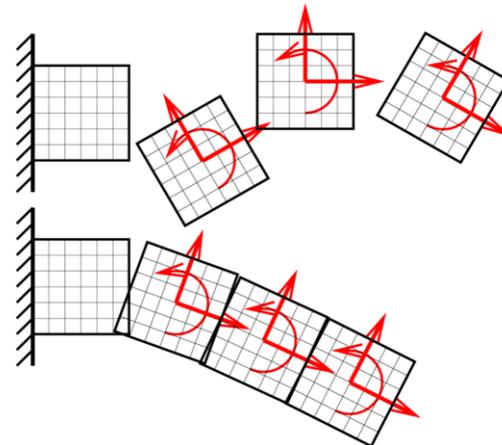
optimal cv

rgb cv

Krylov space methods with domain decomposition

Optimal decomposition for 3D regular grid

Number of DOF = dN^3

Number of interface DOF = $6dN^2 \implies$ complexity of a dot product for FETI iterations

Fill in with sparse direct solver = $O(d^2N^4)$
$\implies$ complexity of a forward-backward substitution

Order of magnitude of $10 < N < 100$

Storage and orthogonalization of many search directions affordable

Use Krylov methods with full orthogonalization
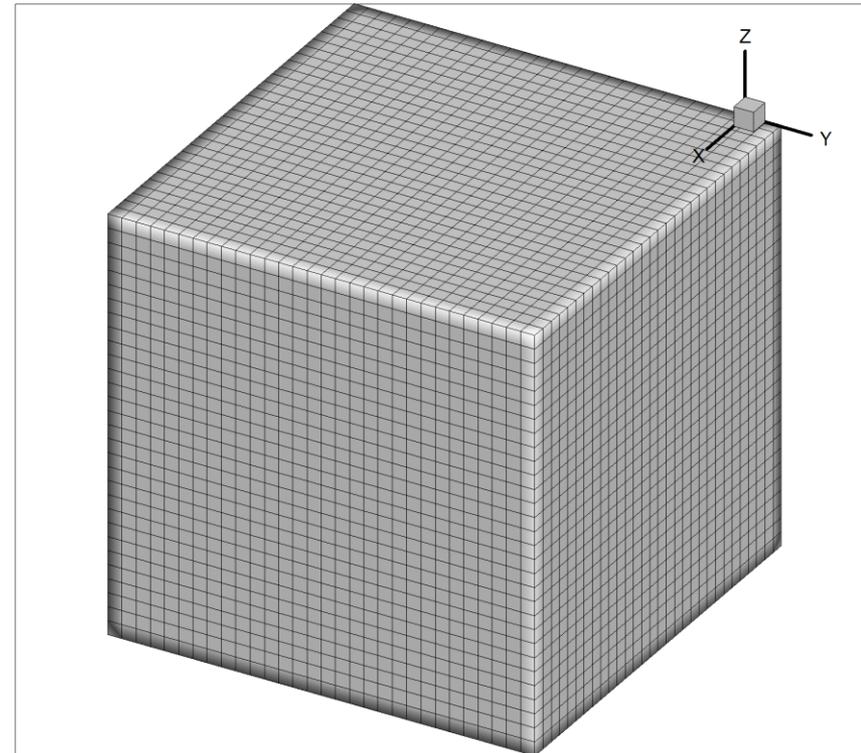
FETI : main features

Very well suited methods for distributed memory system

Good convergence properties (independent on mesh size) for elliptic problems

Need a global "coarse "grid" preconditioner for independence on number of subdomains: built in projector with FETI, generally local constant fields are enough (degree 0 polynomial in subdomain)

Not good for wave propagation problems: Dirichlet and Neumann problems can be ill posed, spurious reflection on interface, local preconditioner not optimal, no null space but many near 0 local frequencies

Robin boundary conditions for wave propagation

Robin problem for Helmholtz

$$
\begin{cases}
-\Delta u - k^2 u = f & \text{in } \Omega \\
\dfrac{\partial u}{\partial n} + \alpha u = g & \text{on } \partial\Omega
\end{cases}
$$

Absorbing conditions: first order approximate "transparent" condition

$$\alpha = \pm jk$$

Well posed problem (no natural frequency)

Variational formulation

$$\int_\Omega \nabla u \,.\, \nabla v - k^2 \int_\Omega uv + \int_{\partial\Omega} \alpha\, uv = \int_\Omega fv + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \qquad \forall v$$

FETI-H for Helmholtz

Augmented matrix associated with inner or outer first order approximate transparent boundary condition

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33}^{(1)} + jkm_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3^{(1)} + \lambda \end{pmatrix} \qquad \begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33}^{(2)} - jkm_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3^{(2)} \end{pmatrix} = \begin{pmatrix} b_2 \\ b_3^{(2)} - \lambda \end{pmatrix}$$

$$K_{33}^{(1)} + jkm_{33} + K_{33}^{(2)} - jkm_{33} = K_{33}$$

Interface residual :

$$x_3^{(1)} - x_3^{(2)} = (S_1 + jkm_{33})^{-1} (\lambda_1 - c_3^{(1)}) - (S_2 - jkm_{33})^{-1}(\lambda_2 - c_3^{(2)})$$
$$= ((S_1 + jkm_{33})^{-1} + (S_2 - jkm_{33})^{-1})\lambda - ((S_1 + jkm_{33})^{-1}c_3^{(1)} - (S_2 - jkm_{33})^{-1}c_3^{(2)})$$
$$= F\lambda - d$$

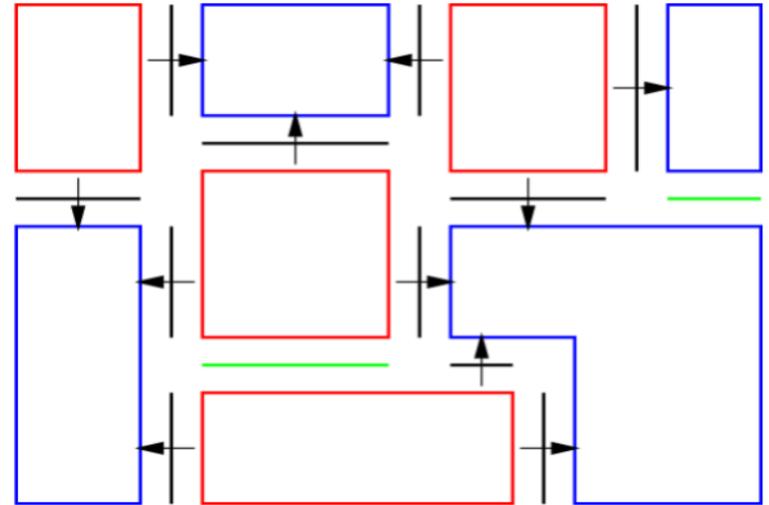F is not SPD anymore, use Orthomin (GCR) or Orthodir instead of CG with full orthogonalization

Well posed local problem if all absorbing conditions are inward or outward

Coloring of subdomains

Mixing of absorbing and non absorbing interfaces

The smaller the number of interfaces without Robin conditions, the better for convergence

Coarse grid preconditioner based on projection

Problem to be solved : $Ax = b$

Coarse space (small dimension) : $Span\{C\}$

Find an optimal correction of approximate solution in $Span\{C\}$

$$g^p = Ax^p - b$$
$$\tilde{x}^p = x^p + C\,\alpha^p$$
$$\tilde{g}^p = g^p + A\,C\,\alpha^p$$
$$\tilde{g}^p \text{ minimal} \Rightarrow (AC)^*\tilde{g}^p = 0$$
$$\tilde{g}^p = \left[\, I - C\,(C^*A^*A\,C\,)^{-1}\,C^*\,\right]\,g^p$$

A^*A -orthogonal projection in the A^*A - orthogonal complement of C

Implementation: compute and factorize

$$(AC)^*(AC)$$

Coarse grid preconditioner for FETI-H

Plane waves in various space directions

The dimension of coarse space must grow with frequency

Worst case: 1-D slicing of wave guide without damping

Coarse grid preconditioner still an open issue for wave propagation

FETI Methods

FETI-2LM

François-Xavier Roux

ONERA and LJLL (retired)

FETI-2LM: independent local Robin conditions

Global system of equations

$$\begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$
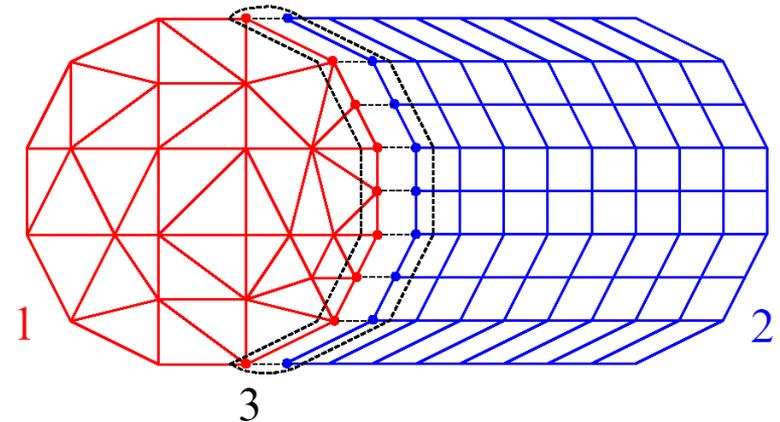
Local systems of equations with Robin boundary conditions

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33}^{(1)} + \alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3^{(1)} + \lambda_1 \end{pmatrix}$$

$$\begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33}^{(2)} + \alpha_2 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3^{(2)} \end{pmatrix} = \begin{pmatrix} b_2 \\ b_3^{(2)} + \lambda_2 \end{pmatrix}$$

Interface interconnecting conditions

$$\begin{cases} x_3^{(1)} = x_3^{(2)} \\ \alpha_1 \, x_3^{(1)} + \alpha_2 \, x_3^{(2)} = \lambda_1 + \lambda_2 \end{cases} \Leftrightarrow$$

$$\begin{cases} \lambda_1 + \lambda_2 - (\alpha_1 + \alpha_2) \, x_3^{(2)} = 0 \\ \lambda_1 + \lambda_2 - (\alpha_2 + \alpha_1) \, x_3^{(1)} = 0 \end{cases}$$

Condensed interface problem

Solution of local systems of equations with Robin boundary conditions

$$(S_1 + \alpha_1)\, \mathrm{x}_3^{(1)} = \lambda_1 + \mathrm{c}_3^{(1)} \qquad\qquad (S_2 + \alpha_2)\, \mathrm{x}_3^{(2)} = \lambda_2 + \mathrm{c}_3^{(2)}$$

Substitution in interface interconnecting conditions : matrix of condensed interface problem

$$\begin{pmatrix} I & I - (\alpha_1 + \alpha_2)(S_2 + \alpha_2)^{-1} \\ I - (\alpha_2 + \alpha_1)(S_1 + \alpha_1)^{-1} & I \end{pmatrix}$$

Optimal interconnecting conditions

Optimal Robin boundary conditions

$$\alpha_1 = S_2 \qquad\qquad \alpha_2 = S_1$$

optimal cv

Optimal Robin boundary conditions = Schur complement of outer domain
The Schur complement is the operator linking the trace of the field and the flux for a field satisfying the equations inside a domain

Relationship with partial block LU factorization : local elimination of local equations in global system

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33} - K_{32}K_{22}^{-1}K_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - K_{32}K_{22}^{-1}\, b_2 \end{pmatrix}$$

$$K_{33} - K_{32}K_{22}^{-1}K_{23} = K_{33}^{(1)} + K_{33}^{(2)} - K_{32}K_{22}^{-1}K_{23}$$

Approximation of Schur complement

Computation of Schur complements in small areas + assembly

Sparse approximation

Black box implementation
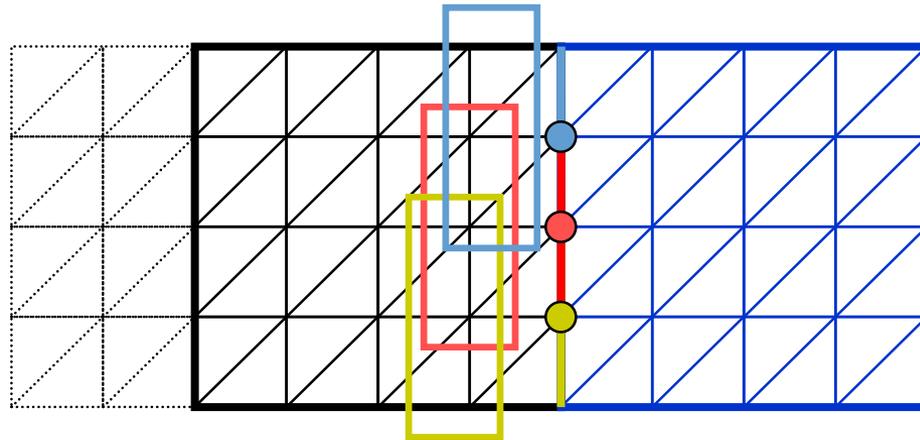
Works Ok for elliptic problems, especially in the case of high heterogeneity accross the interface

Not very robust fore more general problems

Coarse grid preconditioner based on projection

Problem to be solved : $Ax = b$

Coarse space (small dimension) : $Span\{C\}$

Find an optimal correction of approximate solution in $Span\{C\}$

$g^p = Ax^p - b$
$\tilde{x}^p = x^p + C\,\alpha^p$
$\tilde{g}^p = g^p + A\,C\,\alpha^p$
$\tilde{g}^p$ minimal $\Rightarrow$ $(AC)^*\tilde{g}^p = 0$
$\tilde{g}^p = \left[\,I - C\,(C^*A^*A\,C\,)^{-1}\,C^*\,\right]\,g^p$

A^*A -orthogonal projection in the A^*A - orthogonal complement of C

Implementation: compute and factorize

$$(AC)^*(AC)$$

FETI-2LM for wave propagation problems

Dirichlet to Neumann mapping for the exterior problem = exact transparent boundary condition

Discrete Dirichlet to Neumann mapping is dens : need a sparse approximation

First order approximation of transparent condition : $\alpha = jkm$

Higher degree approximations available
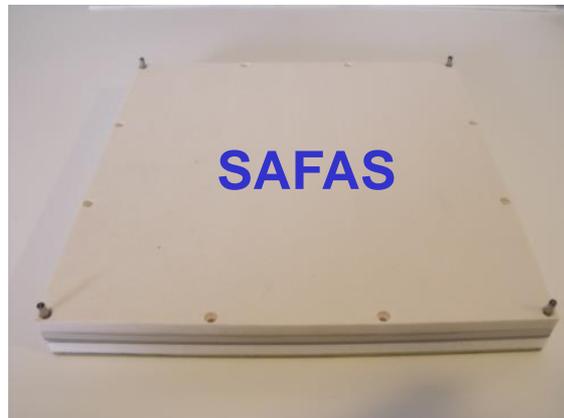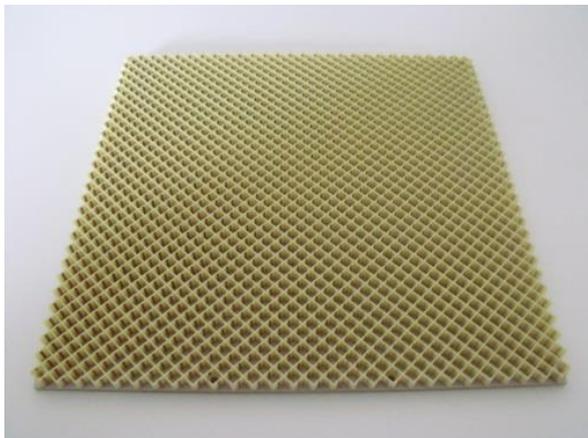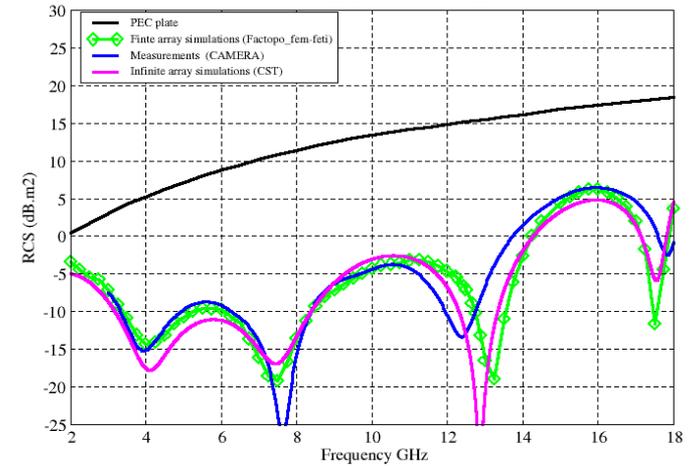
Global coarse grid preconditioner?

rgb cv

Application to electromagnetism

FETI methods

FETI-2LM applied to Maxwell

Robin
$$\begin{cases} \nabla \times (\frac{1}{\vec{\mu}_r} \nabla \times \vec{E}_i) - k_0^2 \vec{\varepsilon}_r \vec{E}_i = k_0^2 (\varepsilon_{r,i} - \mu_{r,i}^{-1}) \vec{E}_{incident} \quad in \qquad \Omega_i \subset R^3 \\ \\ \vec{n}_i \times (\frac{1}{\vec{\mu}_{r,i}} \nabla \times \vec{E}_i) + jk_0 \vec{n}_i \times (\vec{n}_i \times \vec{E}_i) = \vec{\Lambda}^i_{\ j} \quad on \qquad \Gamma_i \qquad (Robin) \\ \\ \\ \vec{n} \times (\nabla \times \vec{E}_i) + jk_0 \vec{n} \times (\vec{n} \times \vec{E}_i) = 0 \quad on \qquad \Gamma_{ext} = \partial\Omega_i \setminus \Gamma_i \end{cases}$$
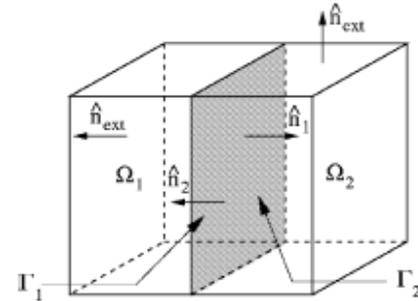
Lagrange multipliers with approximate transparent Robin condition (approximate outer Dirichlet-Neumann)

$$\vec{n}_i \times (\mu_{r,i}^{-1}. \nabla \times \vec{E}_j^i) + jk_0 \vec{n}_i \times (\vec{n}_i \times \vec{E}_j^i) = \vec{\Lambda}_j^i$$

$$\vec{n}_j \times (\mu_{r,j}^{-1}. \nabla \times \vec{E}_i^j) + jk_0 \vec{n}_j \times (\vec{n}_j \times \vec{E}_i^j) = \vec{\Lambda}_i^j$$

Electric and Magnetic field continuity

$$\vec{n}_i \times (\vec{n}_i \times \vec{E}_j^i) = \vec{n}_j \times (\vec{n}_j \times \vec{E}_i^j)$$

$$\vec{n}_i \times (\mu_{r,i}^{-1}. \nabla \times \vec{E}_j^i) = -\vec{n}_j \times (\mu_{r,j}^{-1}. \nabla \times \vec{E}_i^j)$$

$$\Lambda_j^i + \Lambda_i^j - 2jk_0 \vec{n}_i \times (\vec{n}_i \times \vec{E}_j^i) = 0$$

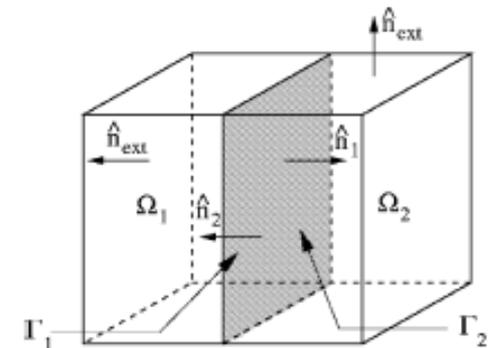$$\Lambda_j^i + \Lambda_i^j - 2jk_0 \vec{n}_j \times (\vec{n}_j \times \vec{E}_i^j) = 0$$

$$\begin{cases} \Lambda_j^i + \Lambda_i^j - 2jk_0\vec{n}_i \times \vec{n}_i \times E_j^i = 0 \\ \Lambda_j^i + \Lambda_i^j - 2jk_0\vec{n}_j \times \vec{n}_j \times E_i^j = 0 \end{cases} \quad on \quad \Gamma_{ij}$$

$$\lambda_j^i + \lambda_i^j - (M_j^i + M_i^j)E_j^i = 0 \quad i = 1,2,..,N_s \quad and \quad j \in neighbor(i)$$

$$M_j^i = jk_0 \int_{\Gamma_{ij}} (\vec{n}_i \times \overrightarrow{W_i}).(\vec{n}_i \times \overrightarrow{W_i})dS$$

Computation of

1. Solution of local problem with Robin conditions defined by λ
2. Exchange values of E and λ on interfaces
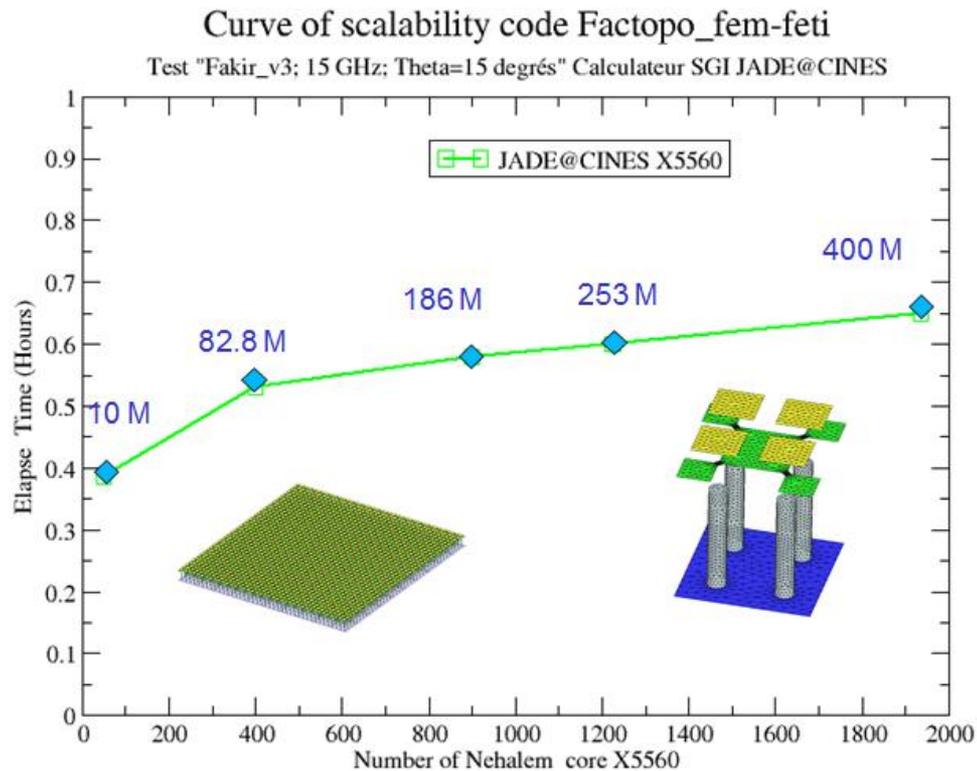3. On each interface Γ_{ij}, computation of :

$$g_i^j = \lambda_j^i + \lambda_i^j - (M_j^i + M_i^j)E_j^i$$

ORTHODIR iterations until || K x - b || < stopping criterion

Weak scalability

Timings with one subdomain per core

Increasing size of the array

Curve of scalability code Factopo_fem-feti

Test "Fakir_v3; 15 GHz; Theta=15 degrés" Calculateur SGI JADE@CINES

ORTHODIR algorithm

ORTHODIR : build a *F*F*-orthogonal basis of Krylov space

$$\begin{cases} g_0 = F\lambda_0 - d \\ \quad v_0 = g_0 \\ \qquad Fv_0 \\ d_0 = \|Fv_0\| \end{cases}$$

$$\begin{cases} \lambda_p = \lambda_{p-1} + \rho_{p-1}v_{p-1} \\ g_p = g_{p-1} + \rho_{p-1}Fv_{p-1} \\ (Fv_{p-1})^*g_p = 0 \Leftrightarrow d_{p-1}\rho_{p-1} = -(Fv_{p-1})^*g_{p-1} \\ \qquad\qquad\qquad\qquad = -(Fv_{p-1})^*g_0 \end{cases}$$

$$\begin{cases} v_p = Fv_{p-1} + \sum_0^{p-1} \gamma_{ip}v_i \\ Fv_p = FFv_{p-1} + \sum_0^{p-1} \gamma_{ip}Fv_i \\ \gamma_{ip} = -(Fv_i)^*FFv_{p-1} \\ d_p = \|Fv_p\| \end{cases}$$

ORTHODIR : block formulation

$$V^p = \begin{bmatrix} v_0 v_1 \cdots v_{p-1} \end{bmatrix} , \ (FV^p)^*(FV^p) = D^p$$

$$\begin{cases} \lambda_p = \lambda_0 + V^p\rho^p \\ g_p = g_0 + FV^p\rho^p \end{cases} \qquad D^p\rho^p = -(FV^p)^*g_0$$

Restarted ORTHODIR with multiple RHS

V^p and FV^p are given , $(FV^p)^* (FV^p) = D_p$

Optimal starting λ_0^{opt}

$$\begin{cases} \lambda_0^{opt} = \lambda_0 + V^p \rho^p \\ g_0^{opt} = g_0 + FV^p \rho^p \end{cases} \qquad D^p \rho^p = -(FV^p)^* g_0 \Leftrightarrow -(FV^p)^* g_0^{opt} = 0$$

Start new iterations with new search directions F^*F −orthogonal to V^p

F^*F −projected ORTHODIR

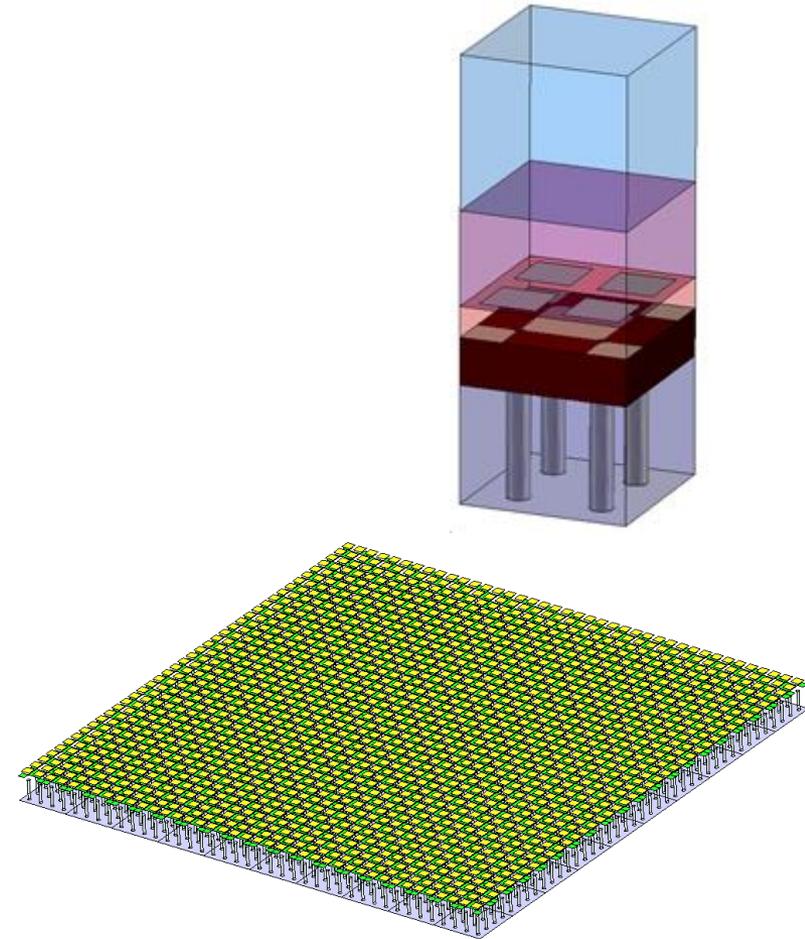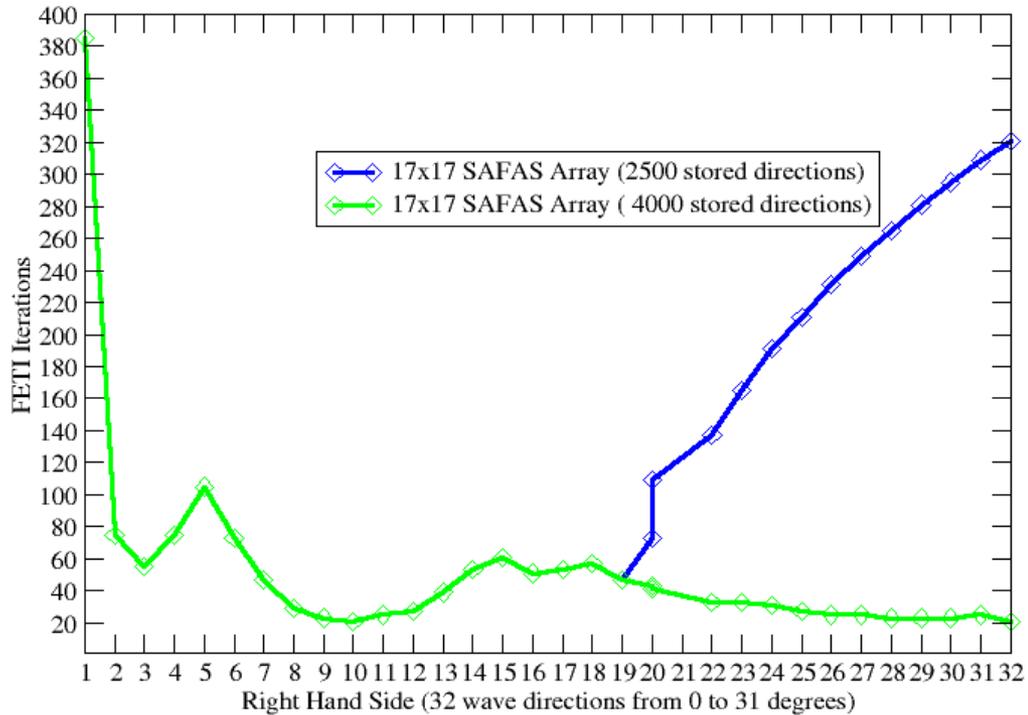In practice same as if restarting ORTHODIR at iteration *p*

Accumulation of search directions with successive RHS

Dependancy upon number of stored directions
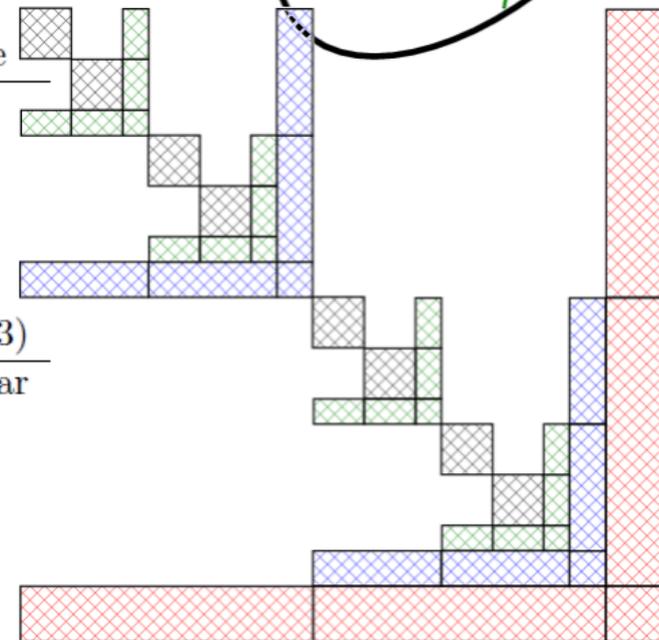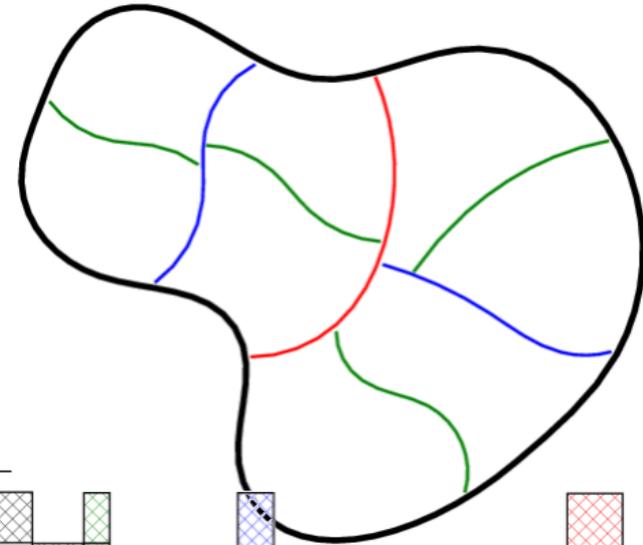
17x17 array

289 subdomains, 50 Million unknowns

31 RHS, incident waves with various angles

Local direct solver on SMP compute node

Sparse direct solver based on nested bisection

P-threads multi-threading management

Splitting of blocks in small sub-blocks

# cores	Dissection		Pardiso	
	CPU time	elapsed time	CPU	esapsed time
1	74.84	72.824	85.04	82.941
2	74.81	38.162	87.79	43.627
4	77.32	20.454	92.66	23.141
6	79.96	15.200	104.38	17.391
8	83.56	12.008	118.25	14.786
12	94.08 ($\times$1.26)	9.873 (/7.38)	165.99 ($\times$1.95)	13.993 (/5.93)

elstct1, $N = 206, 763$, nonsingular

Efficiency of multiple forward-backward substitution

Actual performance limited by global memory access

Parallelization of forward-backward substitution for a single RHS gives very limited performance (speed-up < 2)

With multiple RHS, higher arithmetic complexity with same memory access requirement (better data locality)

	1RHS @ 1core	12 RHS @ 12core	efficiency
Dissection	0.6194 sec.	0.5135 sec	120.6%
Pardiso	0.7054 sec.	1.2642 sec	55.8%

With more than one subdomain per node, memory bandwidth available for each MPI process is even lower

Performance of each single RHS forward-backward substitution is even poorer

Limited parallel efficiency of restarted ORTHODIR

One product by F per iteration

Single RHS local forward-backward substitution

For numerical stability, use modified Gram-Schmidt procedure for orthogonalization

dot product (BLAS1) + global reduction via MPI one by one

Inefficient for local multi-threading on multi-core node

Large communication overhead

Use simultaneous solution

Keep good properties of restarted ORTHODIR

=> block ORTHODIR

Block ORTHODIR algorithm

Block ORTHODIR initialization

$$\begin{cases} g_0^k = F\lambda_0^k - d^k \\ \quad v_0^k = g_0^k \end{cases}, \ k = 1, n_{block}$$

$$\begin{cases} \Lambda_0 = \begin{bmatrix} \lambda_0^1 \lambda_0^2 \cdots \lambda_0^{n_{block}} \end{bmatrix} \\ G_0 = \begin{bmatrix} g_0^1 g_0^2 \cdots g_0^{n_{block}} \end{bmatrix} \\ V_0 = \begin{bmatrix} v_0^1 v_0^2 \cdots v_0^{n_{block}} \end{bmatrix} \\ D_0 = (FV_0)^* (FV_0) = L_0 L_0^* \end{cases}$$

Block ORTHODIR iteration

$$\begin{cases} \Lambda_p = \Lambda_{p-1} + V_{p-1} P_{p-1} \\ G_p = G_{p-1} + FV_{p-1} P_{p-1} \\ FV_{p-1})^* G_p = 0 \Leftrightarrow D_{p-1} P_{p-1} = -(FV_{p-1})^* G_{p-1} \end{cases}$$

$$\begin{cases} V_p = FV_{p-1} + \sum_0^{p-1} V_i \Gamma_{ip} \\ FV_p = FFV_{p-1} + \sum_0^{p-1} FV_i \Gamma_{ip} \\ \Gamma_{ip} = -(FV_i)^* FFV_{p-1} \\ D_p = (FV_p)^* (FV_p) = L_p L_p^* \end{cases}$$

Block Krylov space method implementation for multi-RHS

Optimal solution for each RHS using all search directions computed for all RHS

Rank revealing LL^t factorization of $V^{p^t}AV^p$ or $(AV^p)^*(AV^p)$ block for a set V^p of p candidate search directions, automatic detection of dependencies between search directions, reduction of number of search directions

Same property as recycling in term of decrease of global number of directions to be computed

p simultaneous forward-backward substitutions at each iteration, good parallel efficiency on multi-core nodes

Simultaneous computation of dot products, BLAS3, good parallel efficiency on multi-core nodes, global reduction for a block of scalars at once, reduced MPI overhead

Block recycling strategies straight-forward

Application of multi-RHS techniques

17x17 array, 289 subdomains,
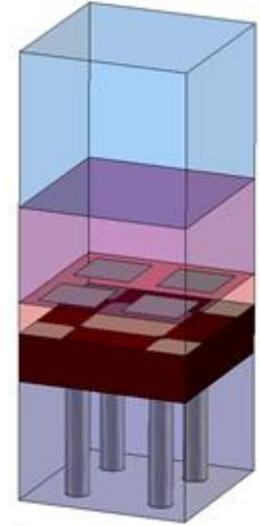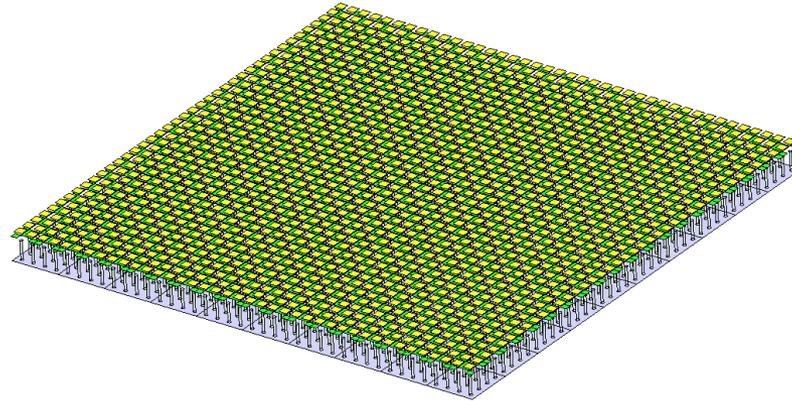**1 core per subdomain,**
50M unknowns,
31 RHS,
16 RHS per block
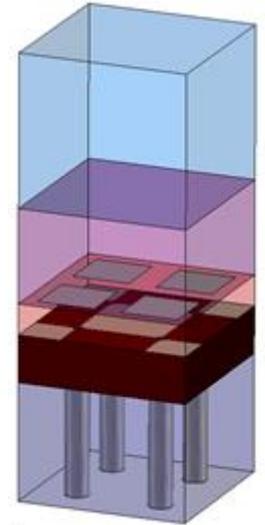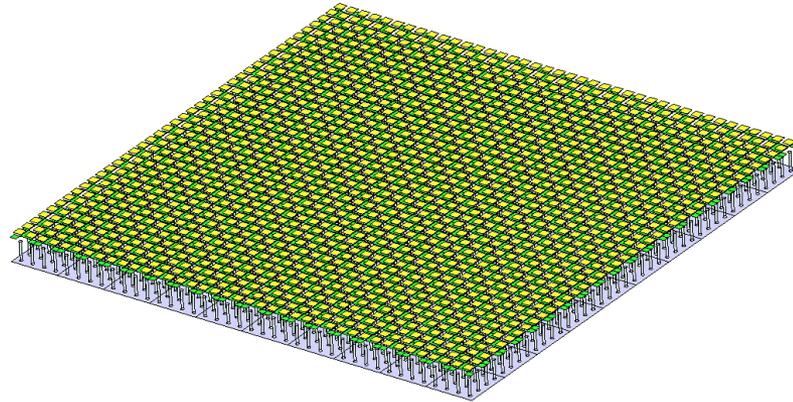
	Elapsed time
Successive ORTHODIR	29.7
Recycling ORTHODIR	3.3
Block ORTHODIR	0.8

Time / 9

Time / 37

Improved local performance with multi-threading

32x1 array, 32 subdomains,
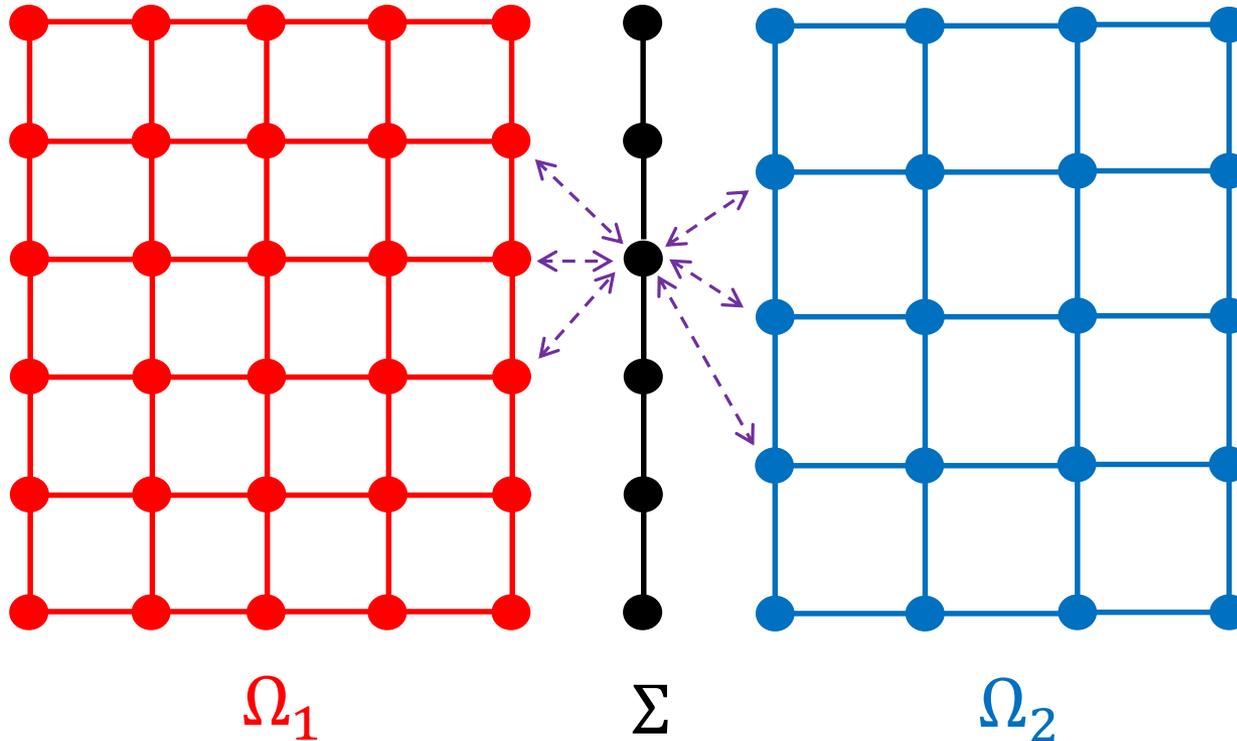**12 cores per subdomain,**
57M unknowns,
32 RHS,
16 RHS per block

	Elapsed time
Successive ORTHODIR	21.8
Continued ORTHODIR	12.1
Block ORTHODIR	0.63

Time / 1.8
Time / 34.6

Limited improvement with continued ORTHODIR in this case, due to right-hand-sides

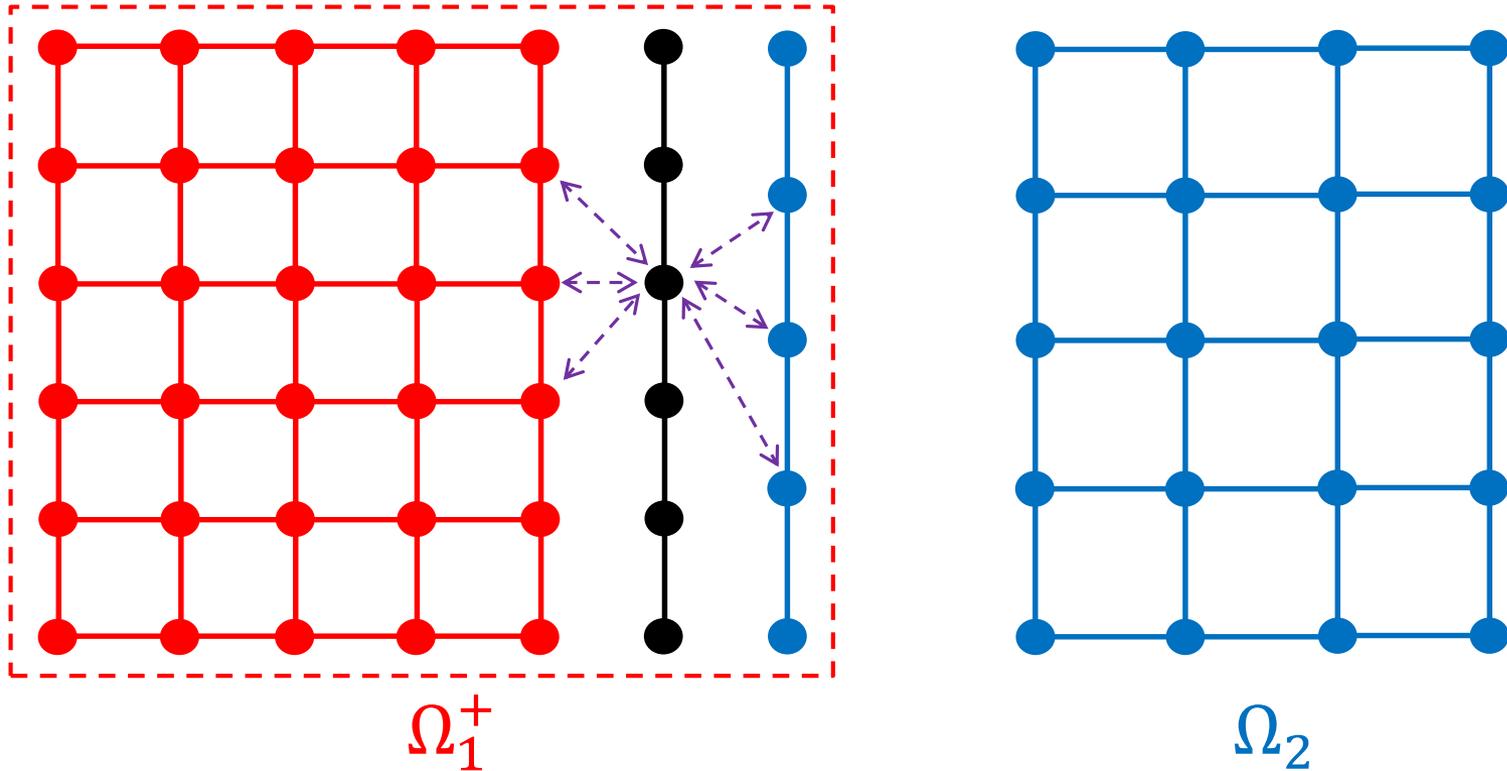$$\Omega_1 \qquad \Sigma \qquad \Omega_2$$

Continuity constraint

$$\int_\Sigma (u_1 - u_2)\mu = 0 \ \ \forall \mu \in W$$

Discretization

$$B_1 \, x_1 - B_2 \, x_2 = M_{11} R_1 x_1 - M_{12} R_2 x_2 = 0$$
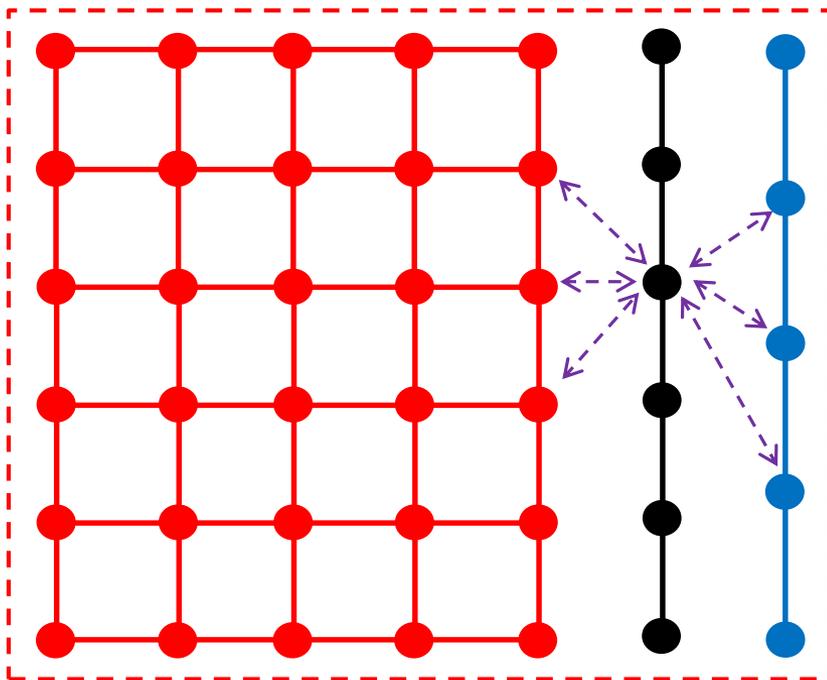
Localization of mortars inside one subdomain

Ω_1^+

Ω_2

Augmented subdomain Ω_1^+ contains mortar nodes plus nodes of Ω_2 interface

Mortar conditions are included in matrix of augmennted subdomain Ω_1^+

Interface between augmented subdomain Ω_1^+ subdomain Ω_2 is conforming

Matrix of augmented subdomain

$$\begin{pmatrix} K_{11} & 0 & R_1^t M_{11} \\ 0 & 0 & -M_{21} \\ M_{11} R_1 & -M_{12} & 0 \end{pmatrix}$$

$K_{11}(N_1, N_1)$ matrix of initial subdomain Ω_1

$R_1(n_1, N_1)$ restriction on interface of Ω_1

$M_{11}(n_1, n_1)$ mass matrix of interface do Ω_1

$M_{12}(n_1, n_2)$, $M_{21}(n_2, n_1)$ coupled mass matrices on interfaces of Ω_1 and Ω_2, $M_{21} = M_{12}^t$

Implementation

Local problem in augmented subdomain Ω_1^+

$$\begin{pmatrix} K_{11} & 0 & R_1^t M_{11} \\ 0 & 0 & -M_{21} \\ M_{11}R_1 & -M_{12} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}$$

dimension of $x_1 = N_1$, dimension of $x_2 = n_2$, dimension of $\lambda = n_1$

Local problem in Ω_1^+ can be ill posed, due to unknowns x_2 , especially if $n_2 > n_1$, but with FETI-2LM and Robin boundary conditions, local matrix is augmented as :

$$\begin{pmatrix} K_{11} & 0 & R_1^t M_{11} \\ 0 & \tilde{S}_{22} & -M_{21} \\ M_{11}R_1 & -M_{12} & 0 \end{pmatrix}$$

Well posed problem with consistant behavior on interface with Ω_2

Similar idea for contact or Discontinuous Galerkin
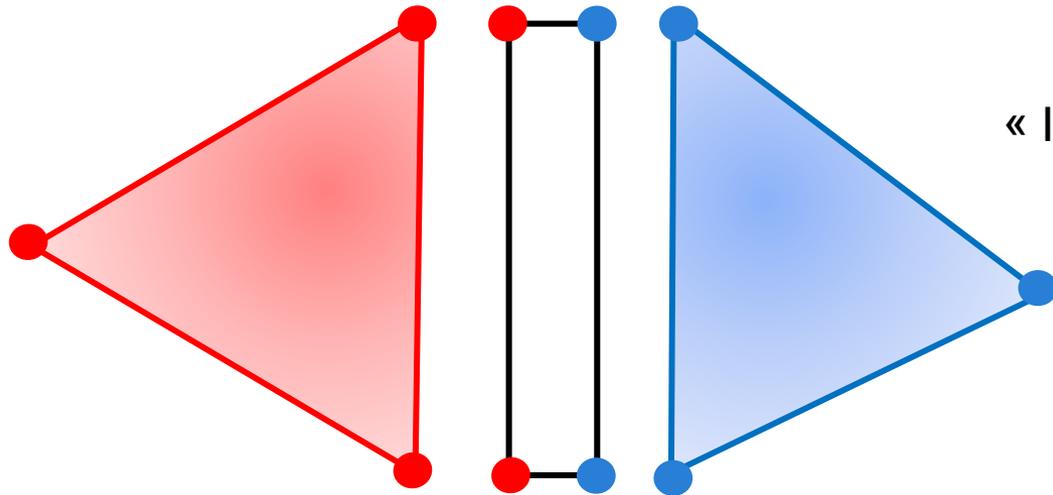
« Interface elements »

Localize all interface
elements in one subdomain