Introduction to multigrid methods. Application of Schwarz domain decomposition methods as smoothers.

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  - Additive
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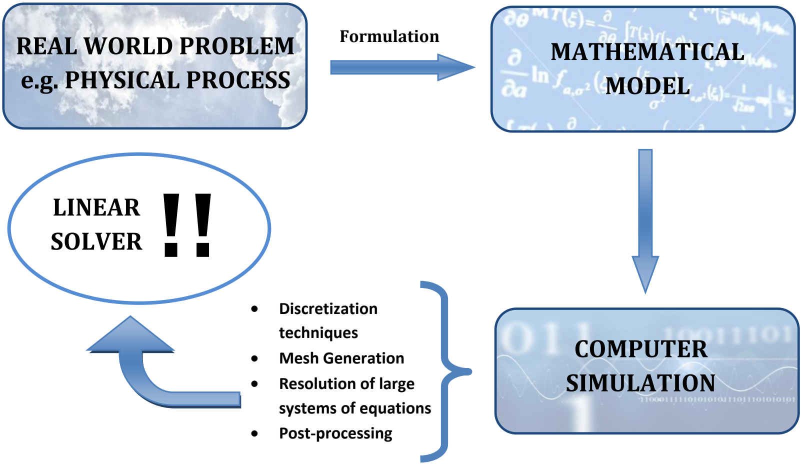
  - High-order discretization for Poisson
  - Saddle-point type problems: Biot's model

Motivation


REAL WORLD PROBLEM
e.g. PHYSICAL PROCESS
Formulation
MATHEMATICAL MODEL
COMPUTER SIMULATION
LINEAR SOLVER !!
• Discretization techniques
• Mesh Generation
• Resolution of large systems of equations
• Post-processing

Numerical Simulation: Solution of systems of equations

NUMERICAL SIMULATION OF PDE MODELS

DISCRETIZATION of PDEs ✓

EFFICIENT SOLUTION OF LARGE SYSTEMS OF EQUATIONS

Desirable properties

- Robust convergence with respect to the discretization and physical parameters.
- Efficiency.

Solution of large systems of equations

Solving linear system of equations:

Given $\boldsymbol{A} \in \mathbb{R}^{N \times N}$ and $\boldsymbol{b} \in \mathbb{R}^{N}$, how to solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ efficiently?

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DIRECT METHODS: Gaussian Elimination, LU

- Black-box, most user-friendly
- Robust, commonly used in practice
- Issue: computational cost is expensive, not efficient!

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$$-\Delta u = f \xrightarrow{\text{FE/FV/FD discretization}} \boldsymbol{Ax} = \boldsymbol{b}$$

Computational cost of Gaussian elimination / LU: $\mathcal{O}(N^3)$

$\boldsymbol{A} \in \mathbb{R}^{N\times N}$	$N = 2^{20}$ $\approx 1.0 \times 10^6$	$N = 2^{24}$ $\approx 1.6 \times 10^7$	$N = 2^{28}$ $\approx 2.6 \times 10^8$	$N = 2^{32}$ $\approx 4.3 \times 10^9$
CPU time	3.8 sec	4.4 hrs	2.0 yrs	83.4 c

Solution of large systems of equations

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Given $\boldsymbol{A} \in \mathbb{R}^{N\times N}$ and $\boldsymbol{b} \in \mathbb{R}^{N}$, how to solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ efficiently?

Efficiency means (nearly) optimal computational complexity
$\mathcal{O}(N \log^{\alpha} N)$, $\alpha = 0, 1$

CLASSICAL ITERATIVE METHODS:

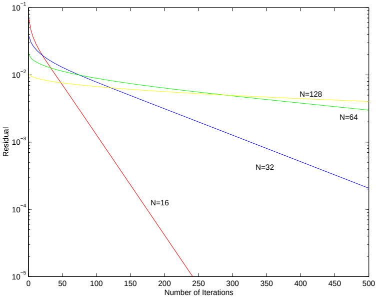
- Jacobi, Gauss-Seidel, etc...
- each iteration is cheap: computational cost is $\mathcal{O}(N)$ (sparse $\boldsymbol{A}$)
- converges if and only if $\rho < 1$
  ($\rho \equiv$ spectral radius of the iteration matrix)

Solution of large systems of equations: GS Convergence.

$$-\Delta u = f \text{ on } [0,1] \times [0,1] \xrightarrow{\text{FE/FV/FD discretization}} \boldsymbol{Ax} = \boldsymbol{b}$$

$$\rho = 1 - \mathcal{O}(h^2)$$

	$N = 16$	$N = 32$	$N = 64$	$N = 128$
it (ρ)	241 (0.9662)	833 (0.9910)	2938 (0.9977)	10405 (0.9994)



Basic linear iterative method

- may converge slowly and the convergence speed may depend on N

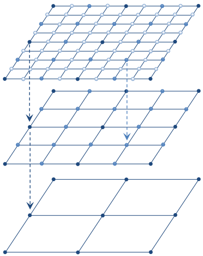
Solution of large systems of equations

Solving linear system of equations:

Given $\boldsymbol{A} \in \mathbb{R}^{N \times N}$ and $\boldsymbol{b} \in \mathbb{R}^{N}$, how to solve $\boldsymbol{Ax} = \boldsymbol{b}$ efficiently?

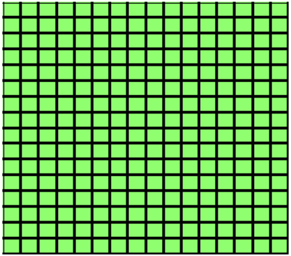
MULTIGRID METHODS

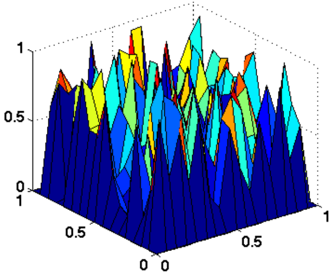
- Robust: Convergence independent of the discretization parameters
- Eficient: Optimal computational cost



Idea of Multigrid

$$-\Delta u = f \text{ on } [0,1]\times[0,1] \xrightarrow{\text{FE/FV/FD discretization}} \boldsymbol{Ax} = \boldsymbol{b}$$





Initial error: $\boldsymbol{e}^0 = \boldsymbol{x} - \boldsymbol{x}^0$

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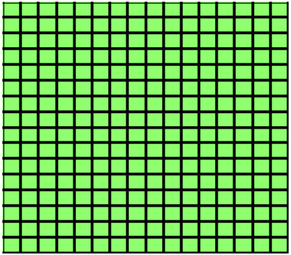

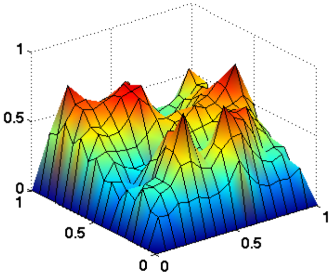

Error after 1 step of Gauss-Seidel

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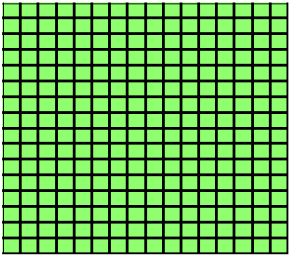

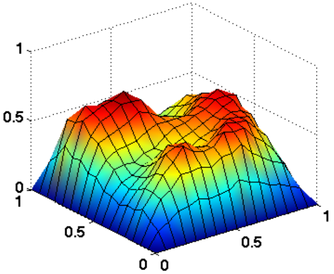

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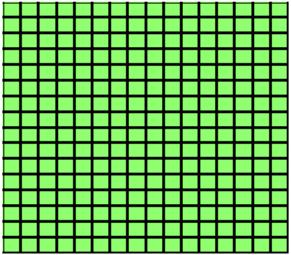

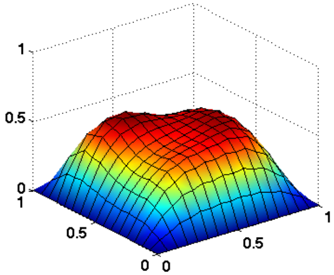

Error after 5 steps of Gauss-Seidel

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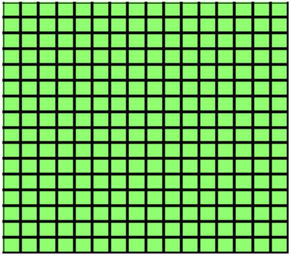

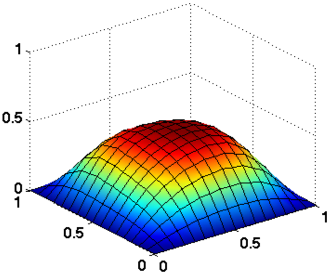

Error after 10 steps Gauss-Seidel

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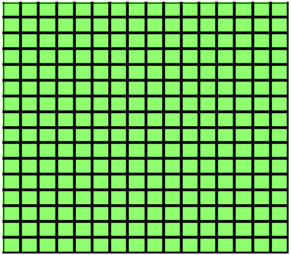

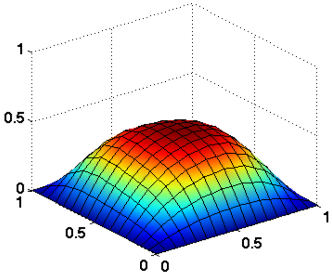

Error after 20 steps Gauss-Seidel

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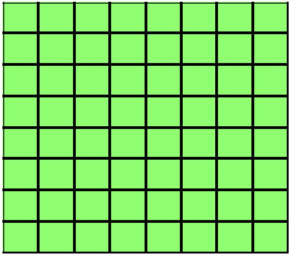

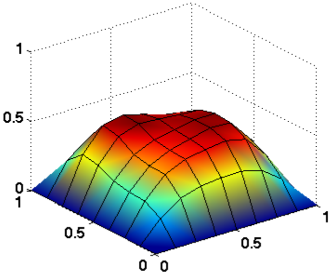

Go to coarse level (after 5 steps Gauss-Seidel)

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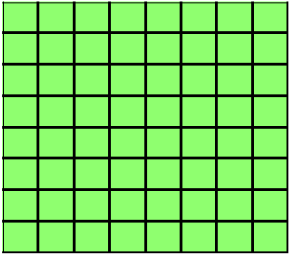

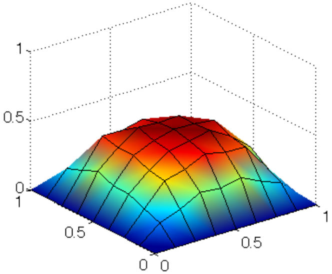

Error after 1 step of Gauss-Seidel on coarse grid

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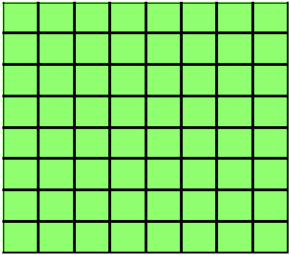

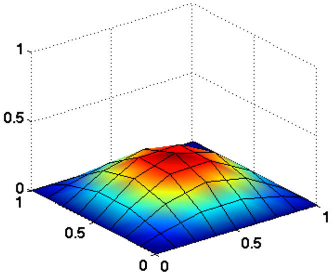

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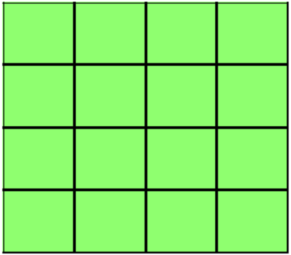

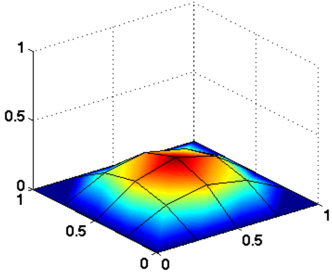

Go to coarser level

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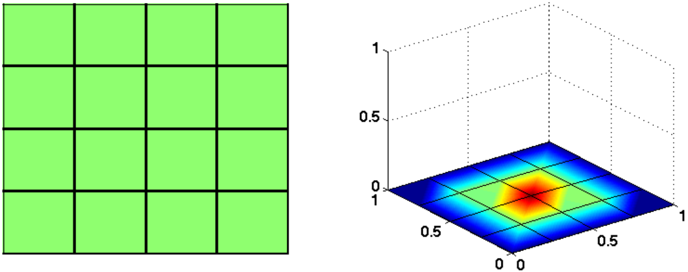
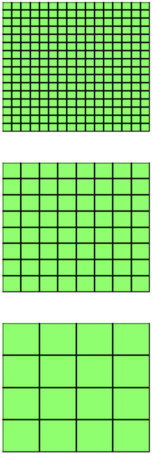


Error after coarse grid correction

Geometric Multigrid Method: $-\Delta u = f \Longrightarrow \boldsymbol{Ax} = \boldsymbol{b}$

Two main components:

- smoothers (Jacobi, Gauss-Seidel, SOR, ...)
- transfer operators (prolongation, restriction)



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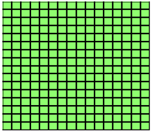

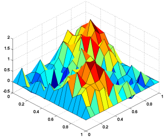

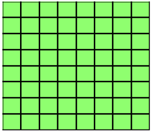

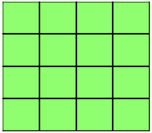

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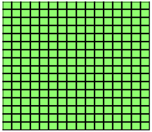

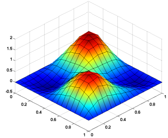

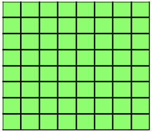

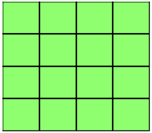

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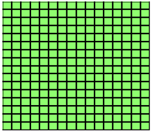

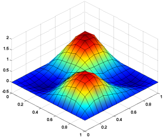

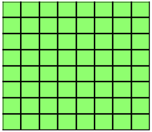

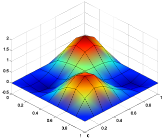

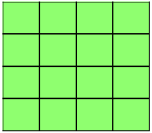

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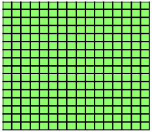

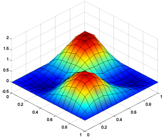

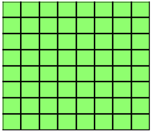

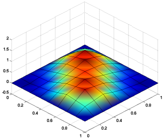

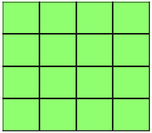

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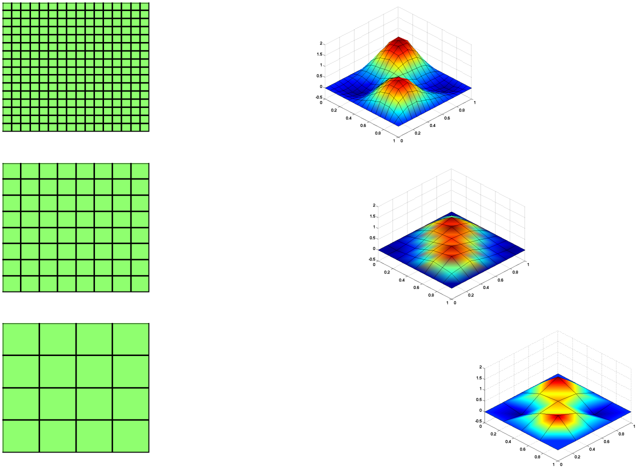

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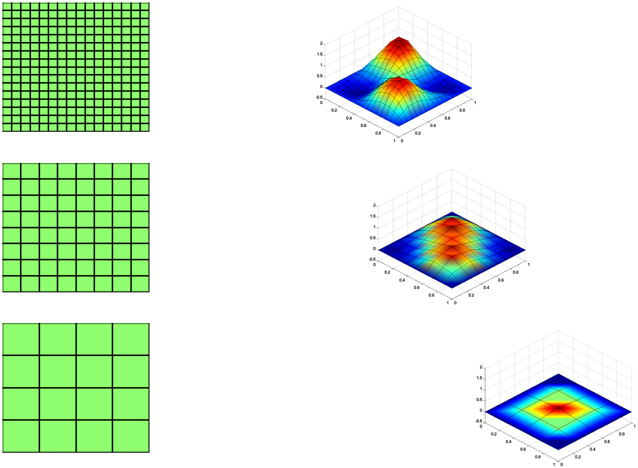

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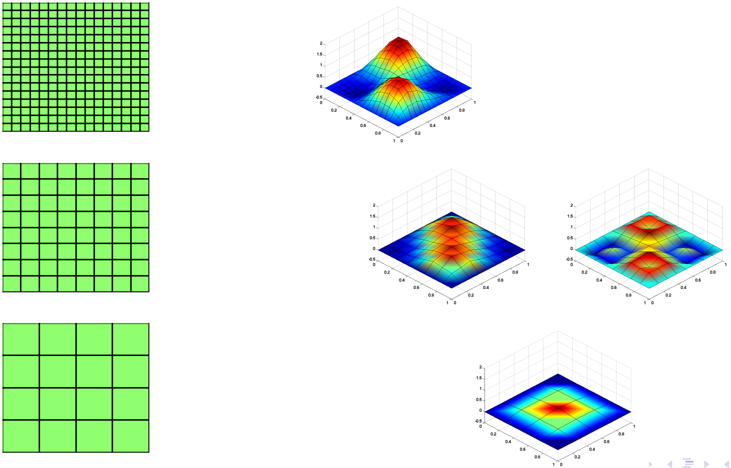

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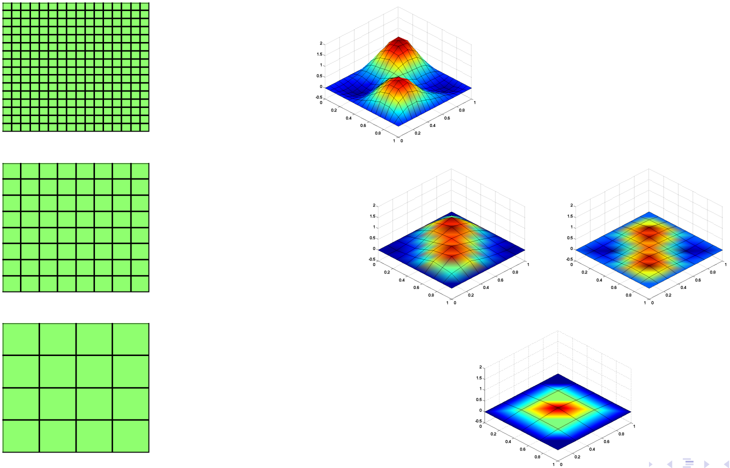

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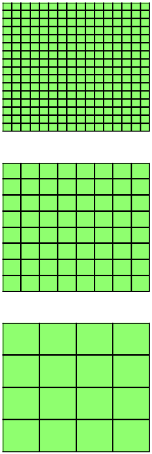

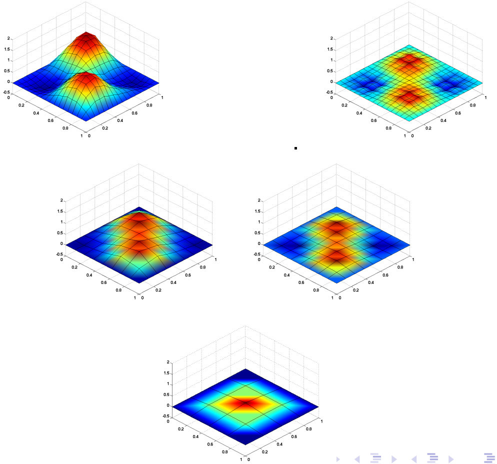

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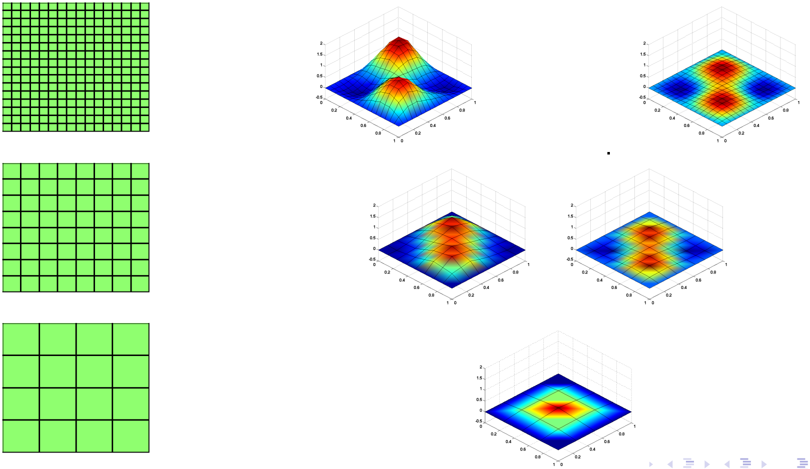


One-dimensional problem

1D boundary value problem: $\begin{cases} -u''(x) = f(x), \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$

Grid: $\Omega_h = \{x_i = ih,\ i = 0, \ldots, n\}, \quad nh = 1.$

Discrete problem: $\begin{cases} -u_{i-1} + 2u_i - u_{i+1} = h^2 f_i,\ 1 \le i \le n-1, \\ u_0 = u_n = 0. \end{cases}$

$$Au = f, \quad \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ \\ \vdots \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} h^2 f_1 \\ \vdots \\ \\ \vdots \\ h^2 f_{n-1} \end{pmatrix}.$$

Weighted Jacobi iteration

- Eigenvalues of A: $\lambda_k(A) = 4\sin^2\left(\frac{k\pi}{2n}\right),\ 1 \le k \le n-1.$
- Eigenvectors of A: $w_j^k = \sin\left(\frac{jk\pi}{n}\right),\ 1 \le k \le n-1,\ 1 \le j \le n-1.$

Consider the weighted Jacobi iteration:

$$u^{j+1} = R_\omega u^j + \omega D^{-1} f, \quad R_\omega = (1-\omega)I + \omega D^{-1}(L+U), \quad A = D - L - U,$$

$$R_\omega = I - \omega D^{-1} A = I - \frac{\omega}{2} A \ \Rightarrow$$

- Eigenvalues of R_ω: $\lambda_k(R_\omega) = 1 - 2\omega\sin^2\left(\frac{k\pi}{2n}\right),\ 1 \le k \le n-1.$
- Eigenvectors of R_ω: $w_j^k = \sin\left(\frac{jk\pi}{n}\right),\ 1 \le k \le n-1,\ 1 \le j \le n-1.$

For $0 < \omega \le 1,\ |\lambda_k(R_\omega)| < 1,\ \lambda_1(R_\omega) = 1 - 2\omega\sin^2\left(\frac{\pi}{2n}\right) \approx 1 - \frac{\omega\pi^2 h^2}{2}.$

Frequency components. Smoothing factor

Let e^0 be the error in an initial guess:

$$e^0 = \sum_{k=1}^{n-1} c_k w^k \quad \Rightarrow \quad e^m = R_\omega^m e^0 = \sum_{k=1}^{n-1} c_k R_\omega^m w^k = \sum_{k=1}^{n-1} c_k \lambda_k^m(R_\omega) w^k.$$

Low-frequency and high-frequency

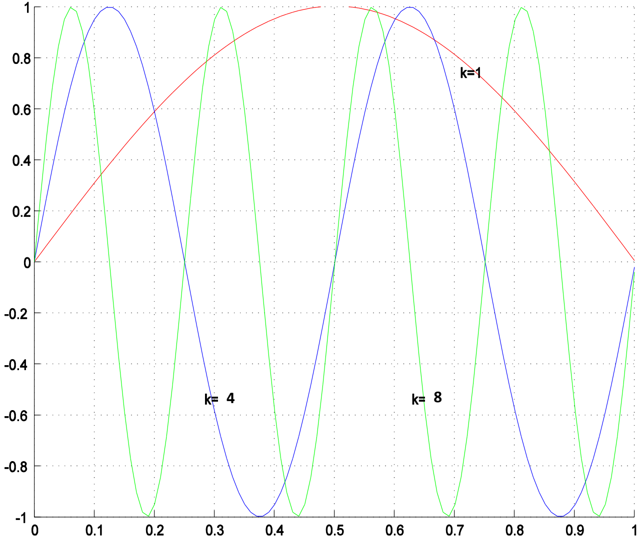
- **Low frequency or smooth mode :** w^k, $1 \leq k < n/2$.
- **High frequency or oscillatory mode:** w^k, $n/2 \leq k \leq n-1$.

Smoothing factor

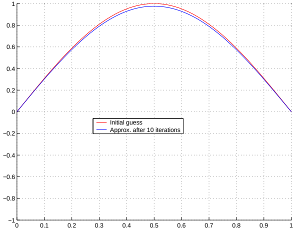
The smoothing factor is the worst factor by which high frequency error components are reduced per relaxation step:

$$\mu(\omega) = \sup\{|\lambda_k(R_\omega)|,\ n/2 \leq k \leq n-1\}.$$

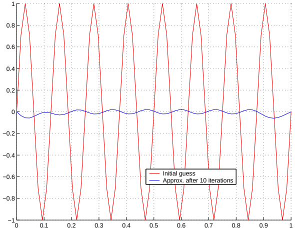
Low and high frequency modes



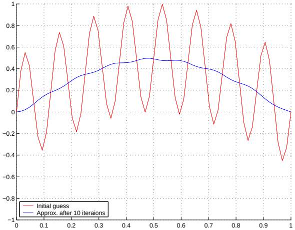
Smoothing property. (n=32).



w^1



w^{16}



$$\frac{w^1 + w^{16}}{2}$$

Frequency components. Smoothing factor

Let e^0 be the error in an initial guess:

$$e^0 = \sum_{k=1}^{n-1} c_k w^k \quad \Rightarrow \quad e^m = R_\omega^m e^0 = \sum_{k=1}^{n-1} c_k R_\omega^m w^k = \sum_{k=1}^{n-1} c_k \lambda_k^m(R_\omega) w^k.$$

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Low-frequency and high-frequency

- **Low frequency or smooth mode :** $w^k, \ 1 \le k < n/2.$
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Smoothing factor

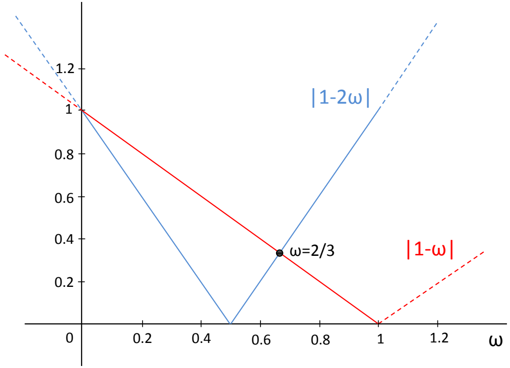
The smoothing factor is the worst factor by which high frequency error components are reduced per relaxation step:

$$\mu(\omega) = \sup\{|\lambda_k(R_\omega)|, \ n/2 \le k \le n-1\}.$$

Smoothing factor for weighted Jacobi

$$\mu(\omega) = \sup\left\{ \left| 1 - 2\omega \sin^2\left(\frac{k\pi}{2n}\right) \right|,\ n/2 \le k \le n-1 \right\}.$$

$$\mu(\omega) = \max\{|1-\omega|, |1-2\omega|\} = \begin{cases} 1, & \omega = 1, \\ 1/3, & \omega = 2/3. \end{cases}$$



Modes on the coarse grid

- Low frequency: kth mode on Ω_h is the kth mode on Ω_{2h}:

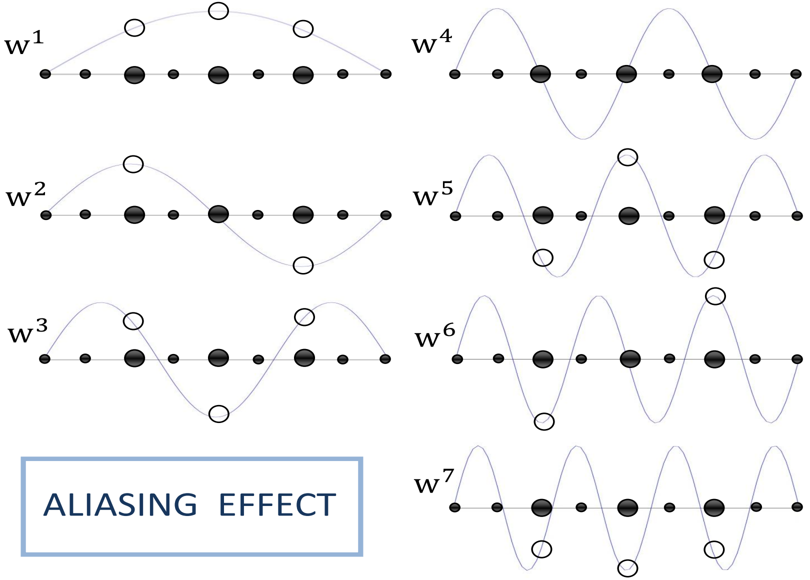
$$w^k_{h,2j} = \sin\left(\frac{2jk\pi}{n}\right) = \sin\left(\frac{jk\pi}{n/2}\right) = w^k_{2h,j}, \quad 1 \le k < n/2.$$

A low-frequency looks more oscillatory on Ω_{2h}.

- The $k = n/2$ mode on Ω_h becomes the zero vector on Ω_{2h}.

- When $k > n/2$: the kth mode on Ω_h becomes the $(n-k)th$ mode on Ω_{2h} $\Rightarrow$ aliasing of frequencies: high frequencies are not visible on the Ω_{2h} grid.

Aliasing of frequencies



ALIASING EFFECT

Coarse grid correction

Discrete linear elliptic boundary value problem: $L_h\, u_h = f_h, \text{ in } \Omega_h$

Let u_h^m be an approximation of the solution u_h, then:

- Error: $e_h^m = u_h - u_h^m$
- Defect or residual: $r_h^m = f_h - L_h\, u_h^m$

Defect equation: $L_h\, e_h^m = r_h^m$

Let Ω_H be a coarser grid $(H > h) \Rightarrow L_H\, \widehat{e_H^m} = r_H^m$

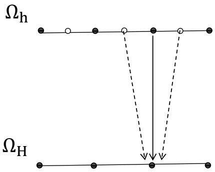
Transfer operators between Ω_h and Ω_H are considered:

- Restriction operator:

$$I_h^H : G(\Omega_h) \to G(\Omega_H)$$

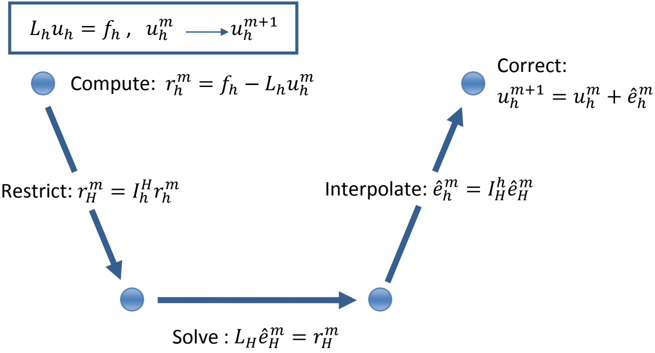
- Interpolation operator:

$$I_H^h : G(\Omega_H) \to G(\Omega_h)$$



Coarse grid correction, $u_h^m \to u_h^{m+1}$

- Compute the defect: $r_h^m = f_h - L_h\, u_h^m$
- Restrict the defect: $r_H^m = I_h^H\, r_h^m$
- Solve exactly the defect equation on Ω_H: $L_H\, \widehat{e}_H^m = r_H^m$
- Interpolate the correction: $\widehat{e}_h^m = I_H^h\, \widehat{e}_H^m$
- Compute a new approximation: $u_h^{m+1} = u_h^m + \widehat{e}_h^m$



Two-grid cycle, $u_h^m \to u_h^{m+1}$

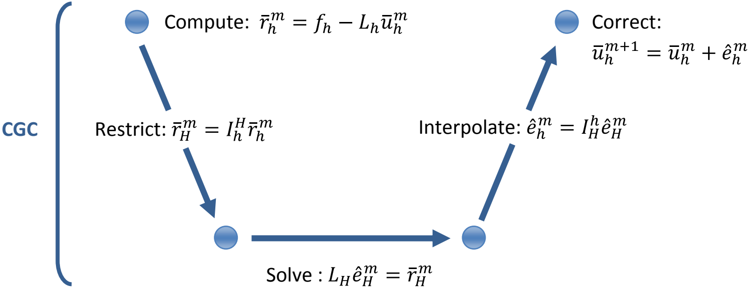
- **Pre-smoothing:**
  - $u_h^m \to \overline{u}_h^m$, by applying ν_1 steps of a given smoothing procedure.
    $$\overline{u}_h^m = S_h^{\nu_1}(u_h^m)$$
- **Coarse grid correction (CGC):**
  - Compute the defect: $\overline{r}_h^m = f_h - L_h \overline{u}_h^m$
  - Restrict the defect: $\overline{r}_H^m = I_h^H \overline{r}_h^m$
  - Solve exactly the defect equation on Ω_H: $L_H \widehat{e}_H^m = \overline{r}_H^m$
  - Interpolate the correction: $\widehat{e}_h^m = I_H^h \widehat{e}_H^m$
  - Compute a new approximation: $\overline{u}_h^{m+1} = \overline{u}_h^m + \widehat{e}_h^m$
- **Post-smoothing:**
  - $\overline{u}_h^{m+1} \to u_h^{m+1}$, by applying ν_2 steps of a given smoothing procedure.
    $$u_h^{m+1} = S_h^{\nu_2}(\overline{u}_h^{m+1})$$

Two-grid cycle

$$L_h u_h = f_h\,, \quad u_h^m \longrightarrow u_h^{m+1}$$

Pre-smoothing (ν_1): $\bar{u}_h^m = S_h^{\nu_1} u_h^m$

Post-smoothing (ν_2): $u_h^{m+1} = S_h^{\nu_2} \bar{u}_h^{m+1}$

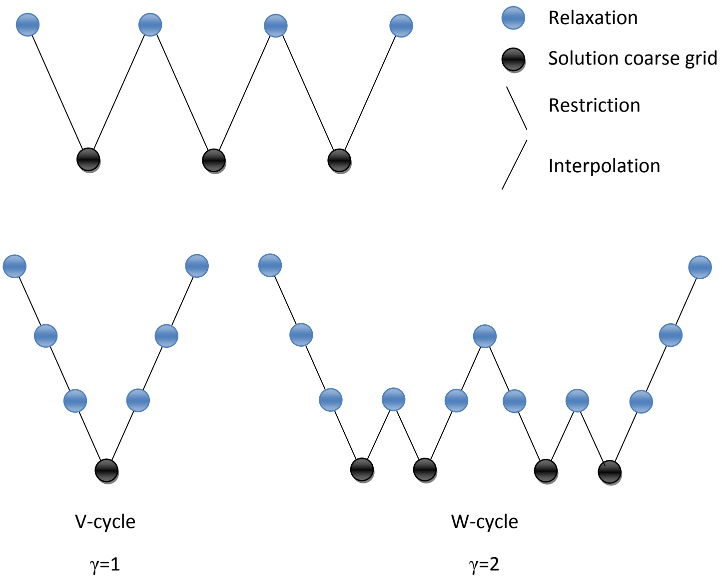


Two-grid iteration operator: $M_h^H = S_h^{\nu_2}[I_h - I_H^h (L_H)^{-1} I_h^H L_h] S_h^{\nu_1}$

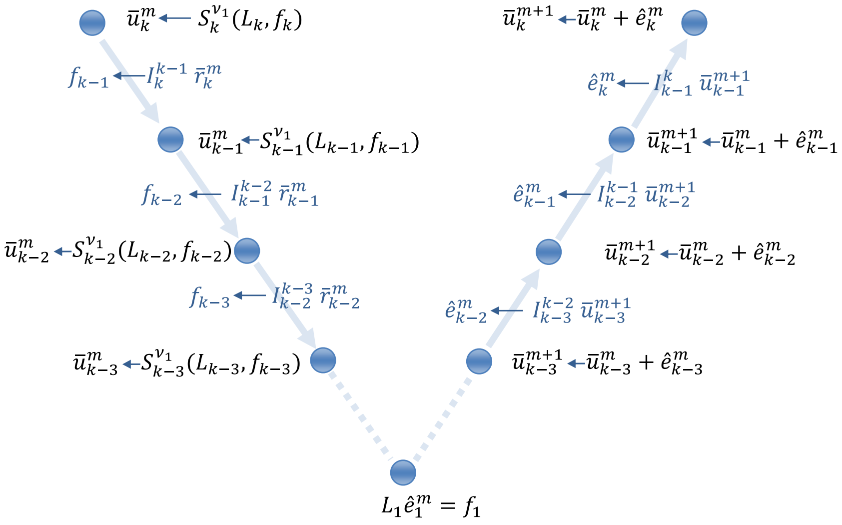
Linear multigrid: $u_k^{m+1} = MG(k, \gamma, L_k, f_k, u_k^m, \nu_1, \nu_2)$

- **Pre-smoothing.** Compute $\bar{u}_k^m$ by applying $\nu_1 (\geq 0)$ smoothing steps to u_k^m.
- **Coarse grid correction:**
  - Compute the defect. $\bar{r}_k^m = f_k - L_k \bar{u}_k^m$.
  - Restrict the defect. $\bar{r}_{k-1}^m = I_k^{k-1} \bar{r}_k^m$.
  - Compute an approximate solution $\widehat{e}_{k-1}^m$ of the defect equation on Ω_{k-1}, $L_{k-1} \widehat{e}_{k-1}^m = \bar{r}_{k-1}^m$ by:
    - If $k = 1$, use a direct or fast iterative solver.
    - If $k > 1$, perform $\gamma \geq 1$ iterations using the zero grid function as a first approximation. $\widehat{e}_{k-1}^m = MG(k-1, \gamma, L_{k-1}, \bar{r}_{k-1}^m, 0_{k-1}, \nu_1, \nu_2)$
  - Interpolate the correction. $\widehat{e}_k^m = I_{k-1}^k \widehat{e}_{k-1}^m$.
  - Correction: $\overline{u}_k^{m+1} = \overline{u}_k^m + \widehat{e}_k^m$.
- **Post-smoothing.** Compute u_k^{m+1} by applying $\nu_2 (\geq 0)$ smoothing steps to $\overline{u}_k^{m+1}$.

Type of cycles in multigrid



V-cycle

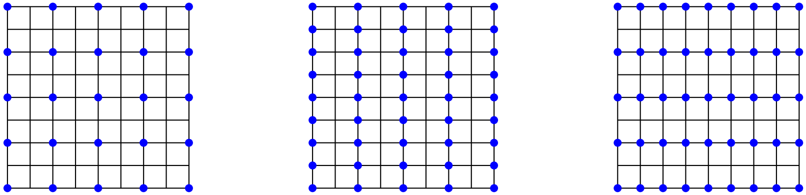

$\bar{u}_k^m \longleftarrow S_k^{\nu_1}(L_k, f_k)$
$f_{k-1} \longleftarrow I_k^{k-1} \bar{r}_k^m$
$\bar{u}_{k-1}^m \leftarrow S_{k-1}^{\nu_1}(L_{k-1}, f_{k-1})$
$f_{k-2} \longleftarrow I_{k-1}^{k-2} \bar{r}_{k-1}^m$
$\bar{u}_{k-2}^m \leftarrow S_{k-2}^{\nu_1}(L_{k-2}, f_{k-2})$
$f_{k-3} \longleftarrow I_{k-2}^{k-3} \bar{r}_{k-2}^m$
$\bar{u}_{k-3}^m \leftarrow S_{k-3}^{\nu_1}(L_{k-3}, f_{k-3})$
$L_1 \hat{e}_1^m = f_1$
$\bar{u}_{k-3}^{m+1} \leftarrow \bar{u}_{k-3}^m + \hat{e}_{k-3}^m$
$\hat{e}_{k-2}^m \longleftarrow I_{k-3}^{k-2} \bar{u}_{k-3}^{m+1}$
$\bar{u}_{k-2}^{m+1} \leftarrow \bar{u}_{k-2}^m + \hat{e}_{k-2}^m$
$\hat{e}_{k-1}^m \longleftarrow I_{k-2}^{k-1} \bar{u}_{k-2}^{m+1}$
$\bar{u}_{k-1}^{m+1} \leftarrow \bar{u}_{k-1}^m + \hat{e}_{k-1}^m$
$\hat{e}_k^m \longleftarrow I_{k-1}^{k} \bar{u}_{k-1}^{m+1}$
$\bar{u}_k^{m+1} \leftarrow \bar{u}_k^m + \hat{e}_k^m$

Components of multigrid

The following individual components of the multigrid method have to be specified:

- The smoothing procedure.
- The numbers ν_1, ν_2 of smoothing steps.
- The coarse grid Ω_H.
- The fine-to-coarse restriction operator I_h^H.
- The coarse grid operator L_H.
- The coarse-to-fine interpolation operator I_H^h.

The coarse grid Ω_H and the coarse grid operator L_H



Standard coarsening | x-semi coarsening | y-semi coarsening

- DCA (Direct coarse aproximation): the direct analog of L_h on the coarse grid Ω_H.
- GCA (Galerkin coarse approximation, $L_H = I_h^H L_h I_H^h$): discontinuous coefficients. Algebraic multigrid methods.

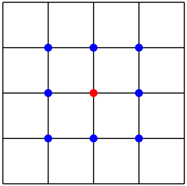
The transfer operators I_h^H and I_H^h

Restriction operator:

Full weighting (FW) operator is frequently used:

We denote the operator by using a stencil [ ] :

$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$



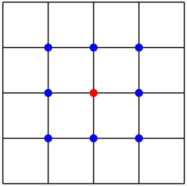
Centered over a coarse point (•) it shows what fractions of the neighboring (•) fine points value is contributed to the value at the coarse point.

Prolongation operator:

Bilinear interpolation is frequently used:

We denote the operator by using a stencil] [:

$$\frac{1}{4}\left]\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix}\right[$$

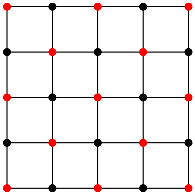


Centered over a coarse-point (•) it shows what fraction of the coarse points value is contributed to neighboring fine points (•).

The smoothing procedure

- Classical iterative methods: Jacobi, Gauss-Seidel, ...
- Take into account:
  - Ordering in which we visit the grid points: Lexicographic order, pattern color, ...
  - Pointwise or blockwise (with or without overlapping) smoother.
  - Damping or relaxation parameter.

Example: Gauss-Seidel Red-Black. (A grid point (x_i, y_j) is red/black if $(i + j)$ is odd/even).



Numerical experiment: Model problem

Model problem: Discrete Poisson equation in 2D with Dirichlet boundary conditions.

Let Ω be the unit square, $\Omega = (0,1)^2 \subset \mathbb{R}^2$.

Model Problem

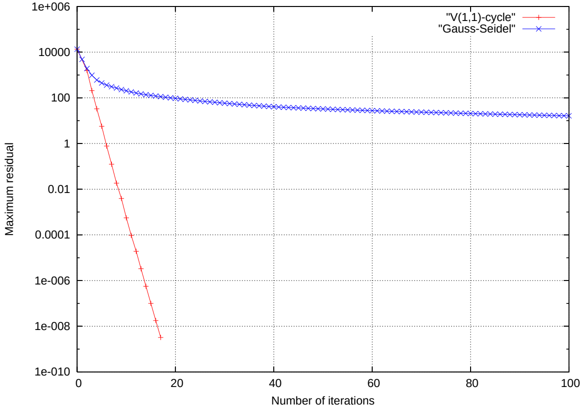
$$\begin{aligned} L_h\, u_h(x,y) &= f_h(x,y), \quad (x,y) \in \Omega_h, \\ u_h(x,y) &= 0, \quad (x,y) \in \partial\Omega_h, \end{aligned}$$

where L_h is the standard 5-point approximation of the operator $-\Delta$, given by

$$L_h\, u_h(x,y) = \frac{1}{h^2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_h u_h(x,y).$$

Numerical experiment: Iterative Methods vs Multigrid

Convergence of a lexicographic Gauss-Seidel against a Multigrid method with the same iterative method as smoother.



Grid: 64×64

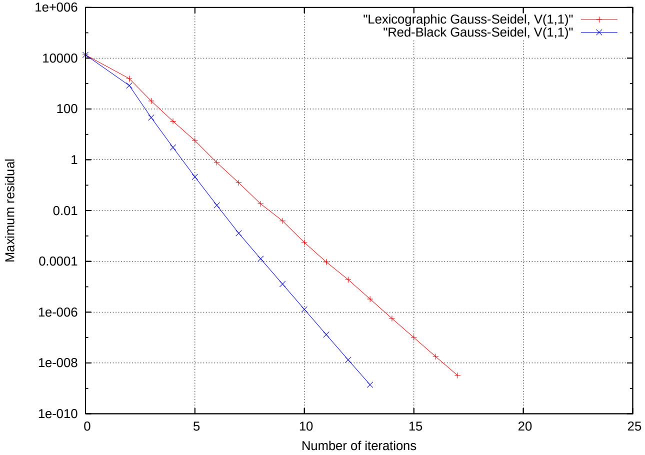
Multigrid features:

- 6 grid levels.
- FW and linear interpolation.
- Smoother: Lexicographic Gauss-Seidel.

Numerical experiment: Influence of the smoother

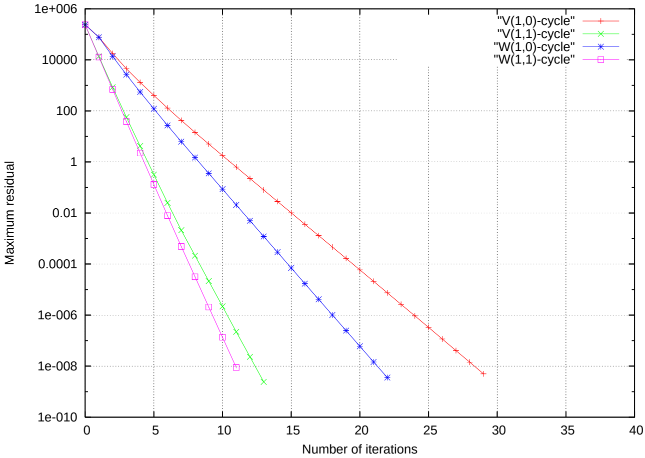
Lexicographic Gauss-Seidel vs Red-Black Gauss-Seidel.

V(1,1)-cycle, Grid: 64×64



Numerical experiment: Influence of the type of cycle

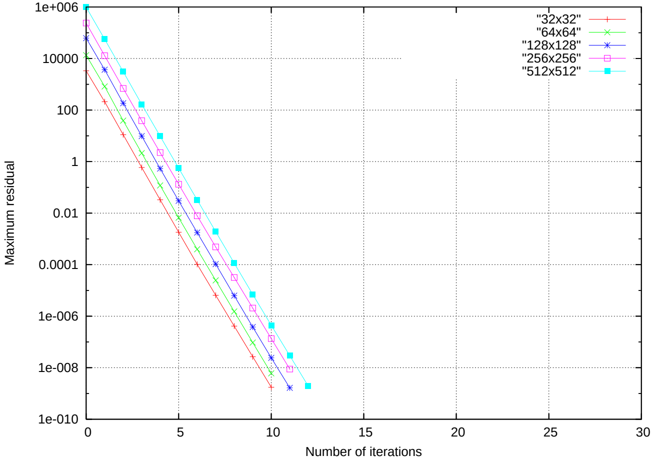
Multigrid convergence for different types of cycles: V-cycle, W-cycle, with different number of pre- and post- smoothing steps.



- Grid: 64×64, (6 grid levels)
- FW and linear interpolation.
- Smoother: Red-Black Gauss-Seidel.

Numerical experiment: h-independent convergence

V(1,1)-multigrid convergence for Red-Black Gauss-Seidel with different numbers of refinement levels.



Numerical experiment: Numerical efficiency of Multigrid

CPU-time and number of iterations necessary to reach a stopping criterion of $\| r^m \| \leq 10^{-10} \| r^0 \|$ with Red-Black smoother, for different cycles and different grids.

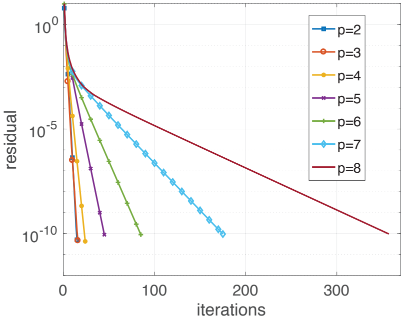
Cycle \ grid	128×128	256×256	512×512	1024×1024
V(1,0)	21 ($< 1''$)	21 ($1''$)	21 ($5''$)	21 ($20''$)
V(1,1)	9 ($< 1''$)	9 ($1''$)	9 ($3''$)	9 ($12''$)
V(2,1)	8 ($< 1''$)	8 ($1''$)	8 ($4''$)	8 ($15''$)
V(2,2)	7 ($< 1''$)	7 ($1''$)	7 ($3''$)	7 ($15''$)
F(1,0)	16 ($< 1''$)	16 ($2''$)	16 ($4''$)	16 ($19''$)
F(1,1)	9 ($< 1''$)	9 ($1''$)	9 ($4''$)	9 ($14''$)
F(2,1)	7 ($< 1''$)	7 ($1''$)	7 ($3''$)	7 ($15''$)
F(2,2)	7 ($< 1''$)	7 ($1''$)	7 ($4''$)	7 ($17''$)

Deterioration of the performance of “standard” multigrid

HIGH-ORDER DISCRETIZATIONS

- Condition number of the system matrix grows exponentially with the polynomial degree
- Naive application of multigrid methods results in an important deterioration of the convergence when the polynomial degree is increased.

Multigrid based
on Gauss-Seidel smoother



- Need of smoothers robust with respect to the polynomial degree

Deterioration of the performance of "standard" multigrid

SADDLE-POINT TYPE PROBLEMS

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}, \text{ where } C = 0 \text{ or } C \text{ is very small.}$$

Some examples:

- Stokes' problem in fluid dynamics
- Porous media problems: Biot's model for example

Point-wise smoothers does not work properly

Block-wise smoothers - Schwarz DD methods

- Splitting of the grid into blocks of unknowns
- One smoothing step consists of solving local problems on each block one-by-one either in a Jacobi-type or Gauss-Seidel-type manner.

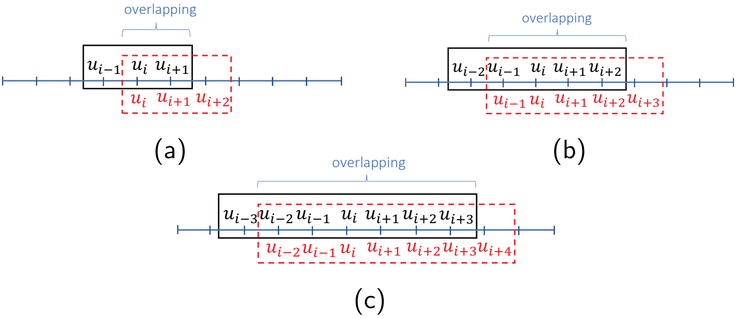


Figure: Splittings of a one-dimensional grid into: (a) blocks of three points, (b) blocks of five points and (c) blocks of seven points, all of them with maximum overlapping among the blocks.

Schwarz domain decomposition methods

- For every block of unknowns B_i, the corresponding local problem is set as follows: Let u_{B_i} be the vector with unknowns involved in block B_i, δu_{B_i} the local corrections and r_{B_i} the local defect. Then, every local system is given by

$$A^{B_i}\delta u_{B_i} = r_{B_i}, \tag{1}$$

  where A^{B_i} is the local matrix corresponding to block B_i.

- The construction of local matrices A^{B_i} is carried out by means of a projection operator V_{B_i} from the vector of unknowns u to the vector of unknowns involved in the block B_i:

$$A^{B_i} = V_{B_i} A V_{B_i}^T.$$

- One iteration of these smoothers consists of a loop over all the blocks such that the corresponding local systems (1) are solved.

Multiplicative Schwarz methods

- Multiplicative Schwarz methods make use of the most recent updates of u.
- For every local system the defect is obtained after a global update of u with the previous local corrections.

Algorithm 1: $u^{m+1} = \text{Multiplicative}(A, u^m, b, M_B, N_B, s_B)$

---

for $i = 1 : N_B$ **do**

$B_i \leftarrow M_B(i, :)$	*Indices of the unknowns involved in B_i.*
$V_i = 0_{s_B \times n}$	
$V_i(:, B_i) = I_{s_B}$	*Construction of projection operator V_i.*
$A^{B_i} \leftarrow V_i A V_i^T$	*Construction of local matrix.*
$r_{B_i} \leftarrow V_i(b - Au^m)$	*Construction of local defect.*
$A^{B_i} \delta u_{B_i} = r_{B_i}$	*Solve the local system.*
$u^{m+1} = u^m + V_i^T \delta u_{B_i}$	*Local correction of u.*

Additive Schwarz methods

- Additive Schwarz methods solve all the local systems by using the same global defect.
- The approximation is corrected at once by addition of all the local corrections.

Algorithm 2: $u^{m+1} = \text{Additive}(A, u^m, b, M_B, N_B, s_B, \omega)$

$r \leftarrow (b - Au^m)$ *Initial residual.*

for $i = 1 : N_B$ **do**

  $B_i \leftarrow M_B(i, :)$ *Indices of the unknowns involved in B_i.*

  $V_i = 0_{s_B \times n}$

  $V_i(:, B_i) = I_{s_B}$ *Construction of projection operator V_i.*

  $A^{B_i} \leftarrow V_i A V_i^T$ *Construction of local matrix.*

  $r_{B_i} \leftarrow V_i r$ *Construction of local defect.*

  $A^{B_i} \delta u_{B_i} = r_{B_i}$ *Solve the local system.*

$u^{m+1} = u^m + \sum_{i=1}^{N_B} V_i^T \omega \, \delta u_{B_i}$ *Global correction of u.*

Restrictive additive Schwarz methods

- Restrictive additive Schwarz methods omit multiple local corrections of the same grid point.

Algorithm 3: $u^{m+1} = \text{Restrictive}(A, u^m, b, M_B, N_B, s_B, \omega, ov)$

$r \leftarrow (b - Au^m)$ *Initial residual.*
for $i = 1 : N_B$ **do**
  $B_i \leftarrow M_B(i, :)$ *Indices of the unknowns involved in B_i.*
  $V_i = 0_{s_B \times n}$
  $\tilde{V}_i = 0_{s_B \times n}$ *Auxiliary projection operator $\tilde{V}_i$.*
  $V_i(:, B_i) = I_{s_B}$ *Construction of projection operator V_i.*
  $\tilde{V}_i(:, B_i(1 : s_B - ov)) = I_{s_B - ov}$ *ov is overlapping among blocks.*
  $A^{B_i} \leftarrow V_i A V_i^T$ *Construction of local matrix.*
  $r_{B_i} \leftarrow V_i r$ *Construction of local defect.*
  $A^{B_i} \delta u_{B_i} = r_{B_i}$ *Solve the local system.*

$u^{m+1} = u^m + \sum_{i=1}^{N_B} \tilde{V}_i^T \omega \, \delta u_{B_i}$ *Global correction of u.*

Poisson equation: Multiplicative Schwarz smoothers

First, let us consider the 2D Poisson problem defined in $\Omega = (0,1)^2$ with dirichlet boundary conditions in $\partial\Omega$:

$$\begin{aligned} -\Delta u &= f, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned}$$

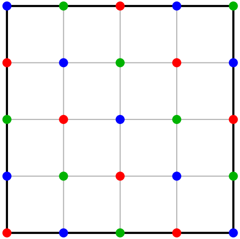
- High-oder finite elements with polynomial degree k.

For this problem, $9(3 \times 3)-$ $25(5 \times 5)-$ and $49(7 \times 7)-$point multiplicative Schwarz smoothers are considered:

	9p Schwarz	25p Schwarz	49p Schwarz
$k = 2$	0.099	0.066	0.050
$k = 3$	0.214	0.067	0.051
$k = 4$	0.455	0.145	0.053
$k = 5$	0.703	0.262	0.119
$k = 6$	0.872	0.440	0.165
$k = 7$	0.955	0.657	0.325
$k = 8$	0.982	0.807	0.440

Poisson equation: Multiplicative Schwarz smoothers

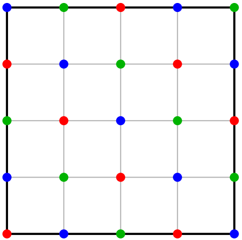
Coloured multiplicative Schwarz relaxations - Improved convergence



Poisson equation: Multiplicative Schwarz smoothers

Coloured multiplicative Schwarz relaxations - Improved convergence

Behavior of the multigrid method based on coloured multiplicative Schwarz smoothers: for a mesh of size 512×512, we display the number of V(1,0)-multigrid iterations together with the CPU times necessary to reduce the initial residual in a factor of 10^{-8}



	9p-Sch		c-9p-Sch		25p-Sch		c-25p-Sch		49p-Sch		c-49p-Sch	
k	it	cpu	it	cpu	it	cpu	it	cpu	it	cpu	it	cpu
2	8	1.29	**4**	**0.68**	7	4.89	4	2.88	6	14.21	3	7.46
3	8	2.13	**4**	**0.99**	6	5.60	4	4.02	6	17.80	3	9.30
4	14	4.66	**7**	**2.40**	7	8.38	4	5.08	6	22.40	4	15.56
5	19	17.26	17	10.40	9	17.83	**4**	**6.09**	6	39.96	3	14.69
6	31	21.53	40	28.40	11	20.01	**5**	**9.33**	7	39.90	3	16.06
7	60	48.04	112	98.70	15	34.32	9	21.80	9	60.44	**3**	**17.15**
8	88	105.62	254	310.55	22	71.19	15	48.87	11	88.17	**4**	**26.01**

Choice of block size depending on the polynomial degree

Poisson equation: Multiplicative Schwarz smoothers

Robust behavior with respect to mesh size h:

Proposed strategy for different grids with mesh-sizes from 128×128 to 1024×1024 and various spline degrees ranging from $k = 2$ to $k = 8$ in order to show the good scalability of the solver. Again, we use $V-$cycles with one pre-smoothing step and no post-smoothing steps

	Color 9p Schwarz						Color 25p Schwarz				Color 49p Schwarz			
	$k = 2$		$k = 3$		$k = 4$		$k = 5$		$k = 6$		$k = 7$		$k = 8$	
Grid	it	cpu	it	cpu	it	cpu	it	cpu	it	cpu	it	cpu	it	cpu
128^2	4	0.06	4	0.10	7	0.22	4	0.62	5	0.99	3	2.22	4	3.35
256^2	4	0.21	4	0.29	7	0.69	4	1.40	5	2.86	3	6.03	4	9.24
512^2	4	0.68	4	0.99	7	2.40	4	6.09	5	9.33	3	17.15	4	26.01
1024^2	4	2.60	4	3.80	7	9.13	3	16.49	5	33.05	3	53.69	4	80.74

Poisson equation: Additive Schwarz smoothers

Additive Schwarz smoother:

Multigrid method based on V(1,1)-cycles using additive Schwarz methods as smoothers with maximum overlapping. In this problem, we consider $9(3 \times 3)$-, $25(5 \times 5)$- and $49(7 \times 7)$-point additive Schwarz smoothers.

	9p ad Schwarz $\omega = 1/9$	25p ad Schwarz $\omega = 1/25$	49p ad Schwarz $\omega = 1/49$
$k = 2$	0.22	0.11	0.06
$k = 3$	0.48	0.27	0.17
$k = 4$	0.72	0.45	0.29
$k = 5$	–	0.62	0.43
$k = 6$	–	0.76	0.56
$k = 7$	–	0.85	0.67
$k = 8$	–	0.96	0.78

Poisson equation: Additive Schwarz smoothers

Element-based Additive Schwarz Smoother

- Size of the blocks is $k = (p+1)^2$
- Minimum overlaping $ov = 1$ is assumed in both directions
- two smoothing steps

	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
AS	0.15	0.21	0.25	0.31	0.36	0.40	0.43
RAS	0.15	0.21	0.25	0.31	0.36	0.40	0.43

Poisson equation: Additive Schwarz smoothers

In order to achieve a determined performance for the solver, we show the asymptotic convergence factors obtained by using $W(1,0)$, $W(1,1)$, $W(2,1)$ and $W(2,2)$ cycles of this approach:

k	$W(1,0)$	$W(1,1)$	$W(2,1)$	$W(2,2)$
2	0.40	0.15	0.06	0.02
3	0.45	0.21	0.09	0.04
4	0.50	0.25	0.13	0.06
5	0.56	0.31	0.18	0.10
6	0.60	0.36	0.21	0.13
7	0.63	0.40	0.25	0.16
8	0.66	0.43	0.28	0.19

Moreover, the asymptotic convergence factors obtained by applying $V-$cycles were substantially similar to the ones obtained with $W-$cycles.

Saddle-Point problem: Biot's model

Equilibrium equation: $\operatorname{div} \boldsymbol{\sigma}' - \alpha \nabla p = \rho \mathbf{g}, \quad \text{in } \Omega,$

Hooke's law: $\boldsymbol{\sigma}' = \lambda \operatorname{tr}(\boldsymbol{\epsilon}) \mathbf{I} + 2\mu \boldsymbol{\epsilon}, \quad \text{in } \Omega,$

Compatibility equation: $\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t), \quad \text{in } \Omega.$

Darcy's law: $$\mathbf{q} = -\frac{\boldsymbol{\kappa}}{\eta_f} \left(\nabla p - \rho_f \mathbf{g} \right), \quad \text{in } \Omega,$$

Continuity equation: $$\frac{\partial}{\partial t} \left(\frac{1}{M} p + \alpha \nabla \cdot \mathbf{u} \right) + \nabla \cdot \mathbf{q} = f, \quad \text{in } \Omega.$$

- λ and μ: Lamé coefficients
- α: Biot-Willis constant and M: Biot modulus
- $\boldsymbol{\kappa}$: Permeability tensor and ρ: bulk density
- η_f: viscosity of the fluid and ρ_f: density of the fluid phase
- $\mathbf{u}$: displacement vector and p: pore pressure
- $\boldsymbol{\sigma}'$ and $\boldsymbol{\epsilon}$: effective stress and strain tensors.
- f: a forced fluid extraction or injection process and g: gravity vector

Saddle-Point problem: Two field formulation for Biot

Two-field (displacement-pressure) formulation

$$
\begin{aligned}
-\nabla(\lambda+\mu)\nabla\cdot\boldsymbol{u} - \nabla\cdot\mu\nabla\boldsymbol{u} + \alpha\,\nabla\,p = \rho\boldsymbol{g},\\
\frac{1}{M}\frac{\partial p}{\partial t} + \alpha\frac{\partial}{\partial t}(\nabla\cdot\mathbf{u}) - \nabla\cdot\left(\frac{1}{\mu_f}K(\nabla p - \rho_f\boldsymbol{g})\right) = f.
\end{aligned}
$$

We complete the formulation of this problem with the following boundary conditions:

$$
\begin{aligned}
\boldsymbol{u} = \mathbf{0}, \quad \frac{K}{\mu_f}(\nabla p - \rho_f\boldsymbol{g})\cdot\boldsymbol{n} = 0, \quad \text{on } \Gamma_c,\\
p = 0, \quad \boldsymbol{\sigma}'\cdot\boldsymbol{n} = \mathbf{0}, \quad \text{on } \Gamma_t,
\end{aligned}
$$

where $\boldsymbol{n}$ is the unit outward normal to the boundary and $\partial\Omega = \Gamma_t \cup \Gamma_c$.
Also, we consider the initial condition

$$
\left(\frac{1}{M}p + \alpha\nabla\cdot\boldsymbol{u}\right)(\boldsymbol{x},0) = 0, \quad \text{on } \boldsymbol{\Omega}.
$$

Saddle-Point problem: Variational formulation

Let us introduce the variational spaces

$$\boldsymbol{V} = \{\boldsymbol{u} \in H^1(\Omega)^n \mid \boldsymbol{u}|_{\Gamma_c} = \boldsymbol{0}\}, \quad Q = \{p \in H^1(\Omega) \mid p|_{\Gamma_t} = 0\}.$$

Then, the variational formulation is given as follows

Variational formulation

For each $t \in (0, T)$, find $(\boldsymbol{u}(t), p(t)) \in \boldsymbol{V} \times Q$ such that:

$$a(\boldsymbol{u}, \boldsymbol{v}) - \alpha(p, \mathrm{div}\boldsymbol{v}) = (\rho \boldsymbol{g}, \boldsymbol{v}), \forall\ \boldsymbol{v} \in \boldsymbol{V},$$

$$\alpha(\mathrm{div}\ \partial_t \boldsymbol{u}, q) + \frac{1}{M}(\partial_t p, q) + b(p, q) = (f, q) + (\boldsymbol{K}\mu_f^{-1}\rho_f \boldsymbol{g}, \nabla q), \forall\ q \in Q$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are given by

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= 2\int_\Omega \mu \epsilon(\boldsymbol{u}) : \epsilon(\boldsymbol{v}) d\Omega + \int_\Omega \lambda \mathrm{div}\boldsymbol{u}\ \mathrm{div}\boldsymbol{v} d\Omega, \\ b(p, q) &= \int_\Omega \frac{\boldsymbol{K}}{\mu_f} \nabla p \cdot \nabla q d\Omega. \end{aligned}$$

Saddle-Point problem: Discrete scheme

- Implicit Euler scheme for discretization in time
- Taylor-Hood elements $(\boldsymbol{V}_h, Q_h)$ for discretization in space.

Fully discretized scheme at time t_m

Find $(\boldsymbol{u}_h^m, p_h^m) \in \boldsymbol{V}_h \times Q_h$ such that

$$a(\boldsymbol{u}_h^m, \boldsymbol{v}_h) - \alpha(p_h^m, \operatorname{div} \boldsymbol{v}_h) = (\rho \boldsymbol{g}, \boldsymbol{v}_h), \quad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

$$\alpha(\operatorname{div} \bar{\partial}_t \boldsymbol{u}_h^m, q_h) + \frac{1}{M}(\bar{\partial}_t p_h^m, q_h) + b(p_h^m, q_h) = (f_h^m, q_h) + (\boldsymbol{K} \mu_f^{-1} \rho_f \boldsymbol{g}, \nabla q_h), \forall\ q_h \in Q_h,$$

where $\bar{\partial}_t \boldsymbol{u}_h^m := (\boldsymbol{u}_h^m - \boldsymbol{u}_h^{m-1})/\tau$, $\bar{\partial}_t p_h^m := (p_h^m - p_h^{m-1})/\tau$,

After discretization, for solving the problem on each time step we have the following saddle point problem

$$\begin{bmatrix} E & B^T \\ -B & \tau C + \frac{1}{M} M_p \end{bmatrix} \begin{bmatrix} \mathbf{u}^m \\ p^m \end{bmatrix} = \begin{bmatrix} g^m \\ f^m \end{bmatrix}.$$

Saddle-Point problem: Schwarz smoother

SCHWARZ TYPE SMOOTHER for Biot:

- Blocks consists of one pressure unknown and all the velocity unknowns that are connected to it.
- Local system of size 51×51 and 76×76 for two-dimensional and three-dimensional problems

High computational cost ➡ Additive Schwarz smoothers

Saddle-Point problem: Schwarz smoother

SCHWARZ TYPE SMOOTHER for Biot:

- Blocks consists of one pressure unknown and all the velocity unknowns that are connected to it.
- Local system of size 51×51 and 76×76 for two-dimensional and three-dimensional problems

High computational cost ➡ Additive Schwarz smoothers

κ	$W(1,0)$	$W(1,1)$	$W(2,1)$	$W(2,2)$
1	0.49	0.22	0.11	0.06
10^{-3}	0.49	0.21	0.11	0.06
10^{-6}	0.49	0.21	0.11	0.06
10^{-9}	0.65	0.42	0.27	0.18
10^{-12}	0.72	0.52	0.56	0.28
10^{-15}	0.72	0.52	0.56	0.28

Table: Asymptotic convergence factors considering natural weights for the $Q2-Q1$ discretization on the unit square domain.

Saddle-Point problem: Schwarz smoother

We can find the optimal weights that provide a robust monolithic multigrid method based on the 51−point additive Schwarz smoother:

- $w_u^{opt} = 0.09$ for displacements at vertices and edges
- $w_u^{opt} = 0.22$ for displacements at interior points
- $w_p^{opt} = 1.02$ for pressure

κ	$W(1,0)$	$W(1,1)$	$W(2,1)$	$W(2,2)$
1	0.58	0.34	0.19	0.11
10^{-3}	0.58	0.34	0.19	0.11
10^{-6}	0.58	0.34	0.19	0.11
10^{-9}	0.58	0.34	0.19	0.11
10^{-12}	0.60	0.36	0.21	0.13
10^{-15}	0.60	0.36	0.21	0.13

Table: Asymptotic convergence factors considering optimal weights w_u^{opt} and w_p^{opt} for the $Q2 - Q1$ discretization on the unit square domain.

Saddle-Point problem: Schwarz smoother

Furthermore, we compared the performance of $V-$cycles and $W-$cycles:

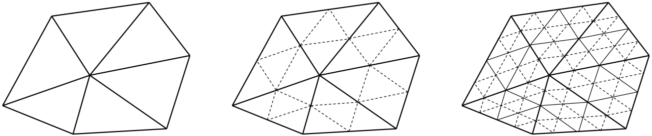
	Smoothing steps (ν_1, ν_2)							
	$(1,0)$		$(1,1)$		$(2,1)$		$(2,2)$	
κ	V	W	V	W	V	W	V	W
1	39	40	20	20	13	14	10	10
10^{-3}	40	40	20	20	14	13	10	10
10^{-6}	39	40	20	20	13	14	10	10
10^{-9}	40	40	23	20	14	13	10	10
10^{-12}	50	43	40	22	18	15	12	12
10^{-15}	50	45	40	23	18	15	12	12

Table: Number of iterations using the optimal weights w_u^{opt}, w_p^{opt}.

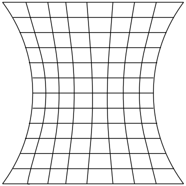
We conclude that $V-$cycles are also robust with respect to κ when applying $V(2,1)$ or $V(2,2)-$cycles yielding a similar number of iterations than $W-$cycles.

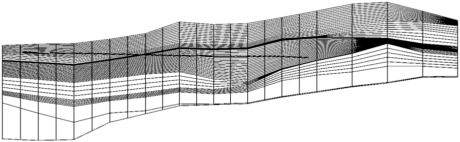
Extension to complex domains

- SEMI-STRUCTURED GRIDS



- LOGICALLY RECTANGULAR GRIDS





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Thank you for your attention!!